ASSIGNMENT PROBLEMS

12.1. INTRODUCTION

As already discussed earlier, linear programming relates to the problems concerning distributions of various resourses (such as *money*, *machines*, *time* etc.), satisfying some constraints which can be algebraically represented as linear *equations/inequalities* so as to *maximize profit* or *minimize cost*. This chapter deals with a very interesting method called the *'Assignment Technique'* which is applicable to a class of very practical problems generally called *'Assignment problems'*.

The name *'Assignment Problem'* originates from the classical problems where the objective is to assign a number of origins (jobs) to the equal number of destinations (persons) at a minimum cost (or maximum profit). To examine the nature of assignment problem, ***suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let c_{ij} be the cost (payment) if the ith person is assigned the jth job, the problem is to find an assignment (which job should be assigned to which person) so that the total cost for performing all jobs is minimum.*** Problems of this kind are known as *assignment problems*.

Table 12·1

Jobs

		1	2	...	j	...	n
	1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}
	2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}
Persons	:	:	:		:		:
	i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}
	:	:	:		:		:
	n	c_{n1}	c_{n2}	...	c_{nj}	...	c_{nn}

Further, such types of problems may consist of assigning men to offices, classes to rooms, drivers to trucks, trucks to delivery routes, or problems to research teams, etc. The assignment problem can be stated in the form of $n \times n$ cost-matrix $[c_{ij}]$ of real number as given in Table 12·1 .

Q. 1. Define Assignment Problem.
2. What is an assignment problem ? **[IGNOU 2001, 99, 97, 96]**

12.2. MATHEMATICAL FORMULATION OF ASSIGNMENT PROBLEM

Mathematically, the assignment problem can be stated as :

Minimize the total cost : $z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$, $i = 1, 2, \ldots, n ; j = 1, 2, \ldots, n$ ...(12·1)

subject to restrictions of the form :

$$x_{ij} = \begin{cases} 1 & \text{if } ith \text{ person is assigned } jth \text{ job} \\ 0 & \text{if not} \end{cases} \quad \ldots(12{\cdot}2)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{(one job is done by the ith person, } i = 1, 2, \ldots, n) \qquad \ldots(12\cdot3)$$

and

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{(only one person should be assigned the jth job, } j = 1, 2, \ldots, n) \qquad \ldots(12\cdot4)$$

where x_{ij} denotes that jth job is to be assigned to the ith person.

This special structure of assignment problem allows a more convenient method of solution in comparison to simplex method.

12·2–1. Assignment Problem as Special Case of Transportation Problem

The assignment problem (as defined in previous chapter) is seen to be the special case of transportation problem when each origin is associated with one and only one destination. In such a case, $m = n$ and the numerical evaluations of such association are called 'effectiveness' instead of 'transportation costs'. Mathematically, all a_i and b_j are unity, and each x_{ij} is limited to one of the two values 0 and 1. In such circumstances, exactly n of the x_{ij} can be non-zero (*i.e.* unity), one for each origin and one for each destination.

Example 1. *A department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours ?*

[JNTU 2002, 2000; Tamil. (ERODE) 97; IAS (Main) 93; Kerala B.Sc. (Math.) 91; Meerut (Stat.) 90; Kalicut B. Tech 90]

Table 12·2

Subordinates

Tasks		I	II	III	IV
	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution. To understand the problem initially, step by step solution procedure is necessary.

Step 1. Subtracting the smallest element in each row from every element of that row, we get the reduced matrix [*Table* 12·3]

Step 2. Next subtract the smallest element in each column from every element of that column to get the second reduced matrix [*Table* 12·4]

Table 12·3

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Table 12·4

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 3. Now, test whether it is possible to make an assignment using only *zeros*. If it is possible, the assignment must be optimal by *Theorem 12.2* of Section 12·3 . Zero assignment is possible in *Table* 12·4 as follows :

(a) Starting with *row* 1 of the matrix (*Table* 12·4), examine the rows one by one until a row containing exactly *single zero* element is found. Then an experimental assignment (indicated by □) is marked to that cell. Now cross all other zeros in the *column* in which the assignment has been made. This eliminates the possibility of marking further assignments in that column. The illustration of this procedure is shown in *Table* 12·5a.

Table 12·5a

	I	II	III	IV
A	[0]	14	9	3
B	9	20	[0]	22
C	23	0	3	0
D	9	12	14	[0]

Table 12·5b

	I	II	III	IV
A	[0]	14	9	3
B	9	20	[0]	22
C	23	[0]	3	0
D	9	12	14	[0]

(b) When the set of rows has been completely examined, an identical procedure is applied sucessively to columns. Starting with *column* 1, examine all columns until a *column* containing exactly one zero is found. Then make an experimental assignment in that position and cross other zeros in the *row* in which the assignment has been made.

Continue these successive operations on rows and columns until all zeros have been either assigned or crossed-out. At this stage, re-examine rows. It is found that no additional assignments are possible. Thus, the complete '*zero assignment*' is given by A → I, B → III, C → II , D → IV as mentioned in *Table 12·5b*. Now According to *Theorem* 12.1, this assignment is also optimal for the original matrix (*Table 12·2*). Now compute the minimum total man-hours as follows :

Optimal assignment	:	A—I	B—III	C—II	D—IV	
Man-hour	:	8	4	19	10	(Total 41 hours.)

Now the question arises : *what would be further steps if the complete optimal assignment after applying Step 3 is not obtained ?* Such difficulty will arise whenever all zeros of any row or column are crossed-out. Following example will make the procedure clear.