Graph Theory

for 6^{th} Semester B.Sc.

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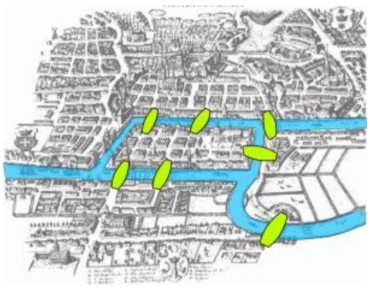
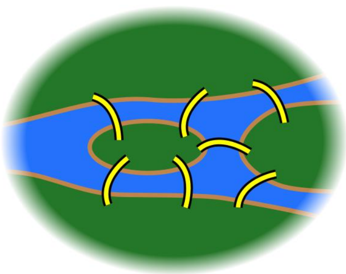
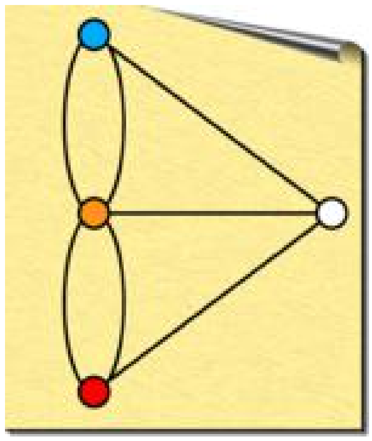
Unit 1

0.1 Did You Know?

0.1.1 The Königsberg bridge problem







The Seven Bridges of Königsberg is a notable historical problem in Mathematics. Its negative resolution by Leonhard Euler in 1735 laid the foundations of Graph theory. The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the main land by seven bridges.
The problem was to find a walk through the city that would cross each bridge only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time (one could not walk half way onto the bridge and then turn around and later cross the other half from the other side).

Euler proved that this problem has no solution.

To start with, Euler pointed out that the choice of route inside each land mass is irrelevant. The only important feature of a route is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms (laying the foundations of Graph theory), eliminating all features except the list of land masses and the bridges connecting them. In modern terms, one replaces each land mass with an abstract "vertex" or node, and each bridge with an abstract connection, an "edge", which only serves to record which pair of vertices (land masses) is connected by that bridge. The resulting mathematical structure is called a graph.

Present state of the bridges

Two of the seven original bridges were destroyed by bombs during World War II. Two others were later demolished and replaced by a modern highway. The three other bridges remain, although only two of them are from Euler's time (one was rebuilt in 1935). Thus, there are now five bridges in Königsberg (modern name Kaliningrad).

Applications

Applications of Graph theory are primarily, but not exclusively, concerned with labelled graphs and various specializations of these. Structures that can be represented as graphs are ubiquitous, and many problems of practical interest can be represented by graphs. The link structure of a website could be represented by a directed graph: the vertices are the web pages available at the website and a directed edge from page A to page B exists if and only if A contains a link to B. A similar approach can be taken to problems in travel, biology, computer chip design, and many other fields. The development of algorithms to handle graphs is therefore of major interest in computer science. There, the transformation of graphs is often formalized and represented by graph rewrite systems. They are either directly used or properties of the rewrite systems(eg. confluence) are studied.

A graph structure can be extended by assigning a weight to each edge of the graph. Graphs with weights, or weighted graphs, are used to represent structures in which pair wise connections have some numerical values. For example, if a graph represents a road network, the weights could represent the length of each road. A digraph with weighted edges in the context of Graph theory is called a network.

Networks have many uses in the practical side of Graph theory, network analysis (eg., to model and analyze traffic networks). Within network analysis, the definition of the term "network" varies, and may often refer to a simple graph.

Many applications of graph theory exist in the form of network analysis. These split broadly into three categories:

1. First, analysis to determine structural properties of a network, such as the distribution of vertex degrees and the diameter of the graph. A vast number of graph measures exist, and the production of useful ones for various domains remains an active area of research.
2. Second, analysis to find a measurable quantity within the network, eg., for a transportation network, the level of vehicular flow within any portion of it.
3. Third, analysis of dynamical properties of networks.

Graph theory is also used to study molecules in chemistry and physics. In condensed matter physics, the three dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. Eg., Franzblau's shortest-path (SP) rings. In chemistry a graph makes a natural model for a molecule, where vertices represent atoms and edges bonds. This approach is especially used in computer processing of molecular structures, ranging from chemical editors to database searching.

Graph theory is also widely used in sociology as a way, eg., to measure actors' prestige or to explore diffusion mechanisms, notably through the use of social network analysis software.

Likewise, Graph theory is useful in biology and conservation efforts where a vertex can represent regions where certain species exist (or habitats) and the edges represent migration paths, or movement between the regions. This information is important when looking at breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species.

0.2 Syllabus

6th SEMESTER
BSCMTC 359 PAPER 8
(Special Paper – 8a)
GRAPH THEORY

UNIT – 1:

Definition of a graph, Konigsberg bridge problem. Finite and infinite graphs, incidence and degree, isolated vertex, pendant vertex and null graph, isomorphism, sub graphs, walks, paths, circuits, connected graphs, components. Euler graphs, operation on graphs. Euler graph. Hamiltonian paths and circuits. Trees: properties, pendant vertices, Distance and centre, rooted and binary tree. Spanning trees, Fundamental circuits.

UNIT – 2:

Cut sets, properties, cut sets in a graph. Fundamental cut sets and circuits. Connectivity and separability. Kurotowski's two graphs. Different representation of planar graphs, Detection of planarity, Geometrical dual.

UNIT – 3:

Vector spaces of a graph: Sets with one operation, with two operations. Modular arithmetic and Galois fields – Recapitulation. Vectors and vector spaces, vector space associated with a graph. Basis vectors of a graph. Circuit and cut set subspaces, orthogonal vectors and spaces. Incidence matrix, submatrices of A(G), Circuit matrix. Fundamental circuit and rank. Cut set matrix. Path matrix, adjacent of matrix.

UNIT – 4:

Chromatic number, chromatic partitioning, chromatic polynomial coverings.

UNIT – 5:

Directed graphs, definition, types of digraph, binary relations and directed paths and connectedness. Euler digraphs, trees and digraphs. Fundamental circuits in digraphs, matrices A, B, C of digraphs, adjacency matrix of a digraph.

A graph consists of vertex set (whose elements are called vertices or points or nodes or junctions) and edge set (whose elements are called edges or lines or arcs or branches), such that each edge is identified with an unordered pair of vertices .

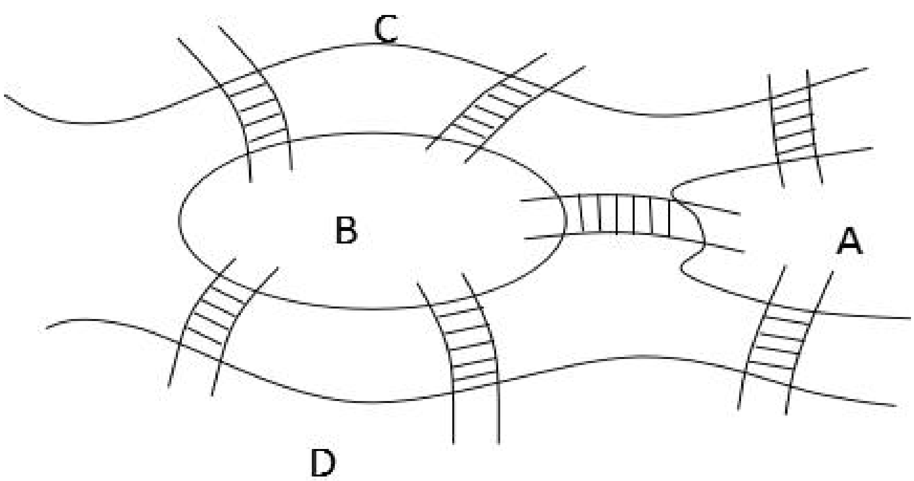
0.3 Definitions

Definition 0.3.1 — GRAPH. A graph $G = (V, E)$ consists of vertex set $V = \{v_1, v_2, v_3, ...\}$ (whose elements are called **vertices** or **points** or **nodes** or **junctions**) and edge set $E = \{e_1, e_2, e_3, ...\}$ (whose elements are called **edges** or **lines** or **arcs** or **branches**), such that each edge e_k is identified with an unordered pair of vertices (v_i, v_j).

0.4 Remarks

- The vertices, v_i, v_j associated with an edge e_k, are called the **end vertices** of e_k.
- The vertex set V of G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$.
- The most common representation of a graph by means of a diagram, in which the vertices are represented as points (dots) and each edge as a line segment joining its end vertices.
- If a vertex v_i is associated with an edge e_k, then e_k is said to be incident on v_i.
- Every edge is incident on two vertices [they are the end vertices of the edge].
- A linear graph means a graph.
- There is no non-linear graph.

0.5 Konigsberg Bridge Problem

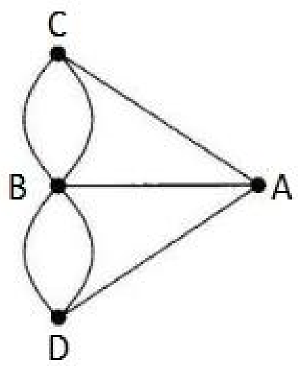


In 1736, Euler wrote the first paper in graph theory. Two islands C and D, formed by the Pregel river in Konigsberg city (was the capital of East Prussia, USSR but now renamed Kaliningrad of West Soviet Russia), were connected to each other and to the banks A and B with seven bridges as shown in the diagram. The problem was to start at any of the 4 land areas of the city (A, B, C or D), walk over each of the 7 bridges exactly once and return to the starting point.

R The Konigsberg Bridge Problem is often said to have been birth of graph theory.

R The originator of the Graph Theory – Leonhard Euler (1707 – 1783).

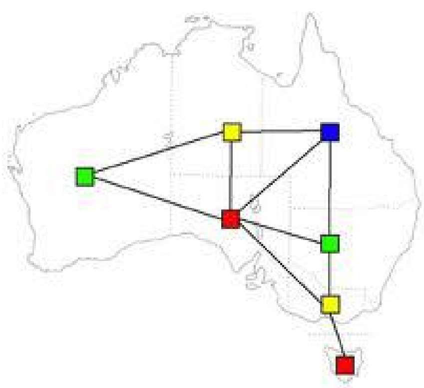
Euler proved that this problem has no solution. He has replaced each land area by a point and each bridge by a line joining these points as follows: The Konigsberg bridge problem is same as the problem of drawing the adjacent figure without lifting the pen/pencil from the paper and without retracing any line and return to the starting point.



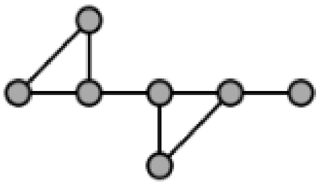
0.6 Four Colour Problem

(posed in 1852, solution found in 1976): states that any map on a plane or on the surface of a sphere can be coloured with four colours in such a way that no two adjacent countries have the same colour.

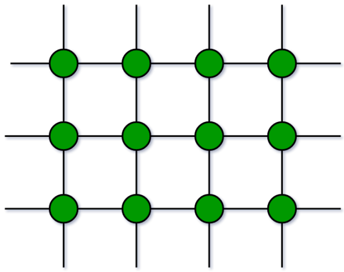
This problem can be translated as a graph theory problem, by representing each country as a point and join the points by a line if the countries are adjacent. The problem is to colour the points in such a way that adjacent points have different colours.



Definition 0.6.1 — FINITE AND INFINITE GRAPHS. A graph is said to be finite if it has finite number of vertices and finite number of edges, otherwise it is called an infinite graph.



Finite graph with 7 vertices



Infinite graph

Definition 0.6.2 — SELF LOOP. If an edge is associated with a vertex pair (v_i, v_i), then it is called a self loop.

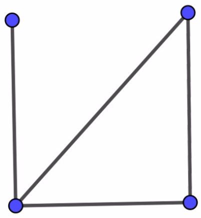
Definition 0.6.3 — PARALLEL EDGES. If two or more edges are associated with same vertex pair, then such edges are called parallel (or multiple) edges.

Definition 0.6.4 — SIMPLE GRAPH. A graph which has neither a self loop nor parallel edges is called a simple graph.

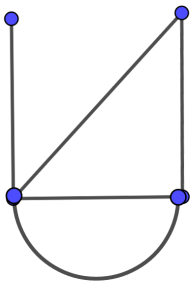
Definition 0.6.5 — MULTIGRAPH. A graph is said to be a multigraph if it contains some parallel edges but no self-loops.

Definition 0.6.6 — PSEUDOGRAPH. A graph is said to be pseudograph if it contains self-loop or parallel edges.

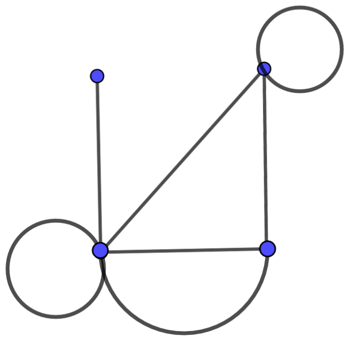
Examples:



Simple graph



Multigraph



Pseudograph

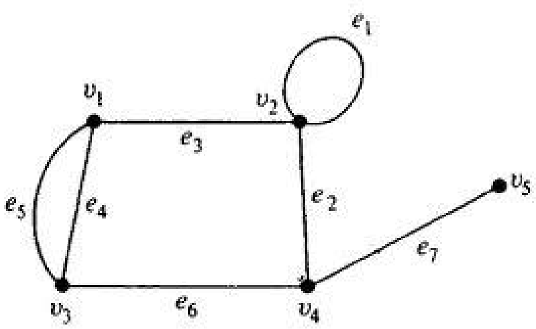
Definition 0.6.7 — INCIDENCE. If v_i is an end vertex of an edge e_k, then v_i and e_k are said to incident on one another.

(R) Two non-parallel edges are said to be adjacent if they are incident on a common vertex.

(R) Two vertices are said to be adjacent if they are the end vertices of the same edge.

Definition 0.6.8 — DEGREE OF A VERTEX. The number of edges incident on a vertex v_i, with self-loop counted twice, is called the degree of the vertex v_i and is denoted by $deg(v_i)$ or $d(v_i)$.

Consider the graph G:



1. e_1 is the self loop.
2. e_4 and e_5 are the parallel edges.
3. e_2 and e_7 are adjacent whereas e_3 and $e_7$are not adjacent.
4. v_2 and v_4 are adjacent whereas v_2 and $v_5$are not adjacent.
5. $deg(v_1) = 3,\ deg(v_2) = 4,\ deg(v_3) = 3,\ deg(v_4) = 3,\ deg(v_5) = 1$

(R) If G is a graph with 'n' vertices and 'e' edges, then $\boxed{\sum_{i=1}^{n} deg(v_i) = 2e}$

Theorem 0.6.1 The number of vertices of odd degree in a graph is always even.

Proof. Let G be a graph with 'n' vertices and 'e' edges. Then, we have $\sum_{i=1}^{n} deg(v_i) = 2e$

$$\Longrightarrow \sum_{\substack{odd\\degree\\vertices}} deg(v_k) + \sum_{\substack{even\\degree\\vertices}} deg(v_j) = 2e$$

$$\Longrightarrow \sum_{\substack{odd\\degree\\vertices}} deg(v_k) = 2e - \sum_{\substack{even\\degree\\vertices}} deg(v_j) = \text{Even number}$$

Since $deg(v_k)$ is odd, the total number of terms in the sum must be even in order that LHS is even.

$\therefore$ There must be an even number of odd vertices in G. ■

Theorem 0.6.2 The maximum number of edges in a simple graph G with n vertices is $\frac{n(n-1)}{2}$.

Proof. Let G be a simple graph with n vertices $v_1, v_2, v_3, ..., v_n$.
Since G is simple graph, we have $d(v_i) \leq (n-1) \quad \forall i$.

$$
\begin{aligned}
2e &= \sum_{i=1}^{n} deg(v_i), \text{ where } e \text{ is the number of edges in } G. \\
2e &\leq \sum_{i=1}^{n} (n-1) \qquad \because d(v_i) \leq (n-1) \\
2e &\leq (n-1)n \\
e &\leq \frac{n(n-1)}{2}
\end{aligned}
$$

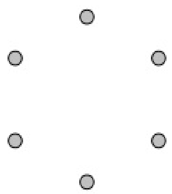
i.e., the maximum number of edges in G is $\frac{n(n-1)}{2}$. ■

Theorem 0.6.3 The maximum degree of any vertex in a simple graph with n vertices is $n-1$.

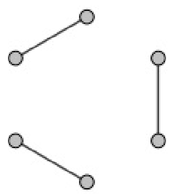
Proof. A simple graph neither contains self loop nor multiple loop.
So, in the worst case it will be connected with all the vertices other than itself in the graph. In that case its degree will be $n-1$. Every other cases will make its degree less than $n-1$.
So the degree of any vertex in a simple graph of n vertices cannot exceed $n-1$.
$\therefore$ The maximum degree of any vertex in a simple graph with n vertices is $n-1$. ■

Definition 0.6.9 — REGULAR GRAPH. A graph, in which all the vertices are of equal degree, is called a regular graph.

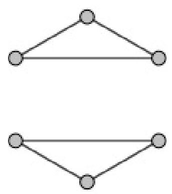
Examples:



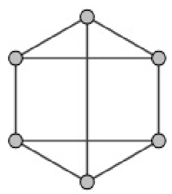
0-regular graph with 6 vertices



1-regular graph with 6 vertices



2-regular graph with 6 vertices



3-regular graph with 6 vertices

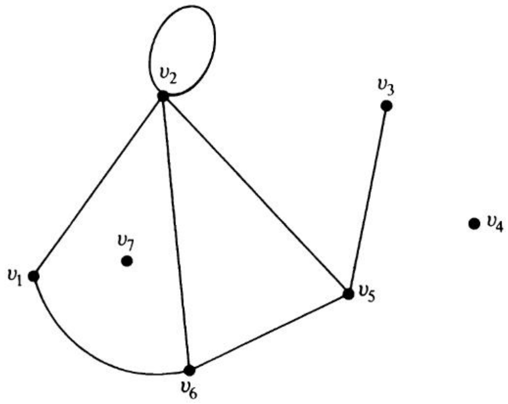
Definition 0.6.10 — ISOLATED VERTEX. A vertex having no incident edge is called an isolated vertex.

R An isolated vertex is a vertex with degree zero.

Definition 0.6.11 — PENDANT VERTEX. A vertex which has only one incident edge on it is called a pendant vertex (or leaf).

R The degree of a pendant vertex is one.

Example: In the following graph, v_4 and v_7 are isolated vertices and v_3 is a pendant vertex.

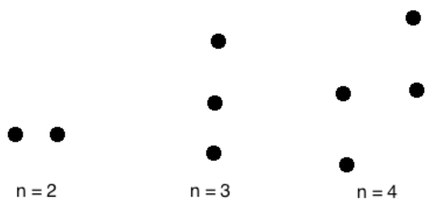


Definition 0.6.12 — EDGES IN SERIES. Two adjacent edges are said to be in series if their common vertex is of degree 2.

Definition 0.6.13 — NULL GRAPH. A graph without any edges is called a null graph.

R Every vertex in a null graph is an isolated vertex.

Examples:

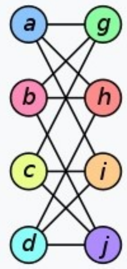
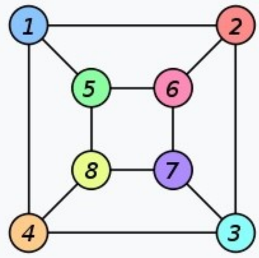


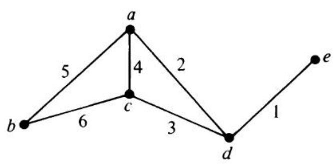
Definition 0.6.14 — ISOMORPHISM. Two graphs G and G' are said to be isomorphic to each other if there is a one-to-one correspondence between their vertices and edges such that the incidence relationship is preserved.

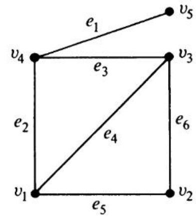
R Two graphs G and G' are said to be isomorphic to each other if G and G' contain

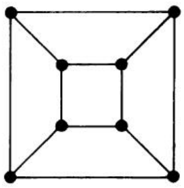
1. equal number of vertices.
2. equal number of edges.
3. equal number of vertices with a given degree.

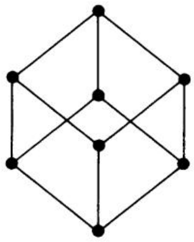
Examples:

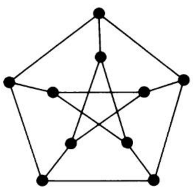
Graph G	Graph H	An isomorphism between G and H
		$f(a) = 1$ $f(b) = 6$ $f(c) = 8$ $f(d) = 3$ $f(g) = 5$ $f(h) = 2$ $f(i) = 4$ $f(j) = 7$

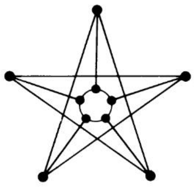


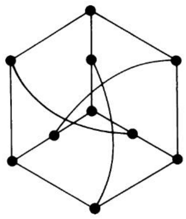


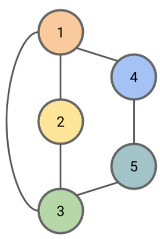


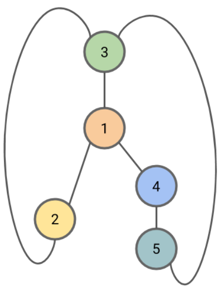


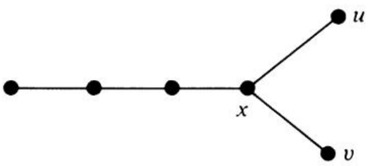


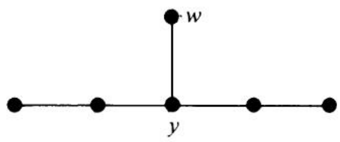








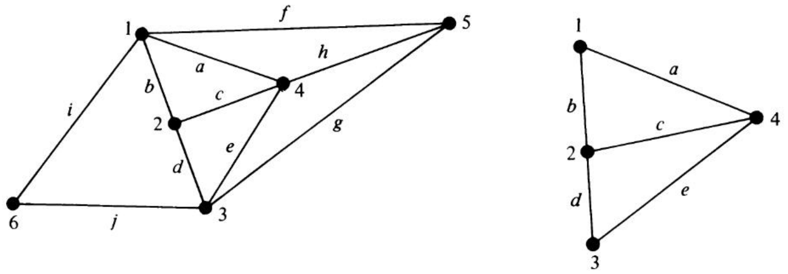


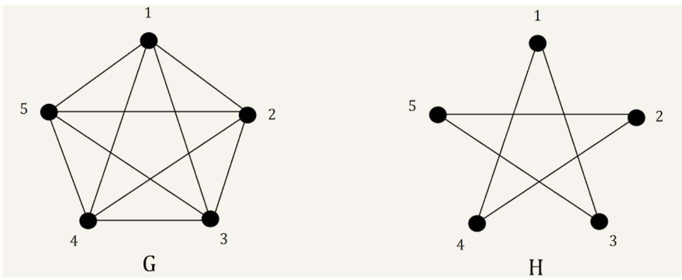


Definition 0.6.15 — SUBGRAPH. A graph g is said to be a subgraph of G if all the vertices and all the edges of g are in G and each edge of g has the same end vertices in g as in G.

R

1. Every graph is its own subgraph.
2. A subgraph of a subgraph of G is a subgraph of G.
3. A single vertex in a graph G is a subgraph of G.
4. A single edge in G, together with its end vertices, is also a subgraph of G.





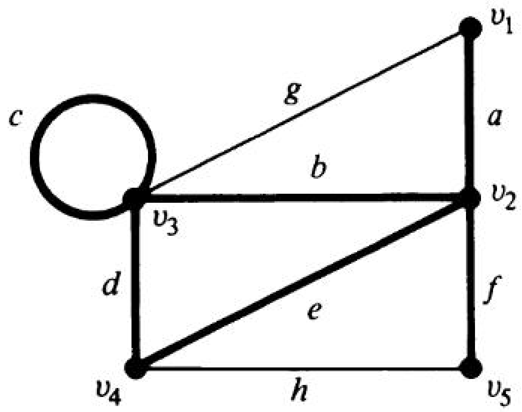
Definition 0.6.16 — WALK. A walk (or edge-train or chain) is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices such that each edge is incident on vertices preceding and following it.

R In a walk, no edge can appear more than once but vertices can appear any number of times.

R A walk which begins and ends in the same vertex is called a closed walk.

R A walk, in which the end vertices are different, is called an open walk.

Example:



In Figure, $v_1\,a\,v_2\,b\,v_3\,c\,v_3\,d\,v_4\,e\,v_2\,f\,v_5$ is a walk.

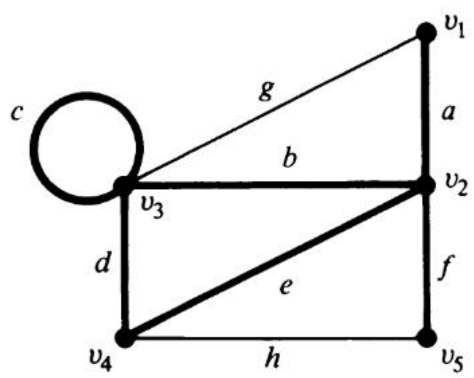
Definition 0.6.17 — PATH. An open walk in which no vertex appears more than once is called a path.

(R) A path does not intersect itself.

(R) The number of edges in a path is called the length of a path.

(R) The terminal vertices of a path are of degree one and the <u>rest of the vertices</u> (are called intermediate vertices) are of degree two.

Example:

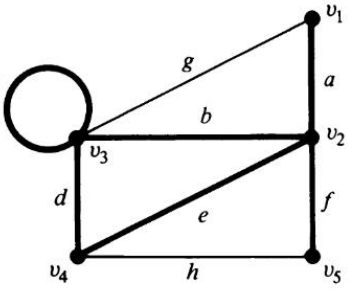


In Figure, $v_1\,a\,v_2\,b\,v_3\,d\,v_4$ is a path, but $v_1\,a\,v_2\,b\,v_3\,c\,v_3\,d\,v_4\,e\,v_2\,f\,v_5$ is not a path.

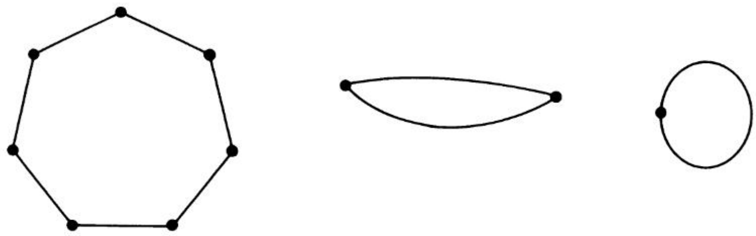
Definition 0.6.18 — CIRCUIT. A closed walk in which no vertex (except the initial and the final vertices) appears more than once is called a circuit.

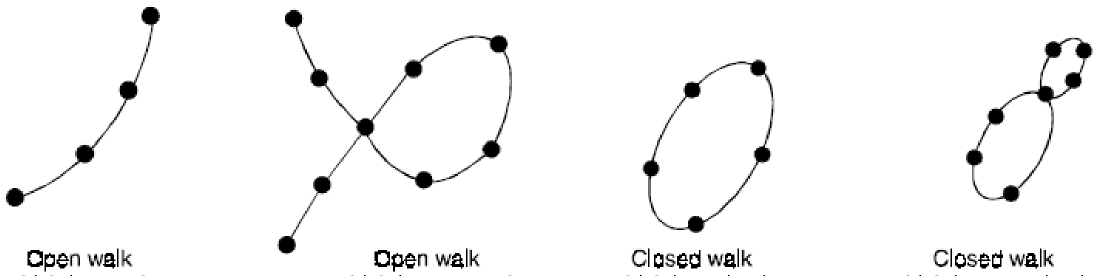
(R) A circuit is a closed, non-intersecting walk.

Examples:



In Figure, $v_2\, b\, v_3\, d\, v_4\, e\, v_2$ is a circuit.

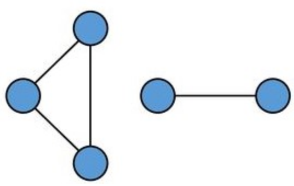




> **Definition 0.6.19 — CONNECTED AND DISCONNECTED GRAPHS.** A graph G is said to be connected if there is at least one path between every pair of vertices in G, otherwise G is disconnected.

R A null graph of more than one vertex is disconnected.

> **Definition 0.6.20 — COMPONENT.** A disconnected graph consists of two or more connected subgraphs. Each of these connected subgraphs is called a component.



Disconnected graph with 2 components

Theorem 0.6.4 A graph is disconnected iff its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2.

Proof. Suppose that such a partitioning exists.
Consider two arbitrary vertices a and b of G, such that $a \in V_1$ and $b \in V_2$.
No path can exist between vertices a and b. Otherwise, there would be at least one edge whose one end vertex would be in V_1 and the other in V_2.
Hence, if a partition exists, G is not connected.
Conversely, let G be a connected graph.
Consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a.
Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form a (non-empty) set V_2. No vertex in V_1 is joined to any vertex in V_2 by an edge. Hence the partition. ■

Theorem 0.6.5 If a graph (connected or disconnected) has exactly two vertices of odd degree, then there must be a path joining these two vertices.

Proof. Let G be a graph with all even vertices except vertices v_1 and v_2, which are of odd degree. Which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices.
$\therefore$ In graph G, the vertices v_1 and v_2 must belong to the same component and hence must have a path between them. ■

Theorem 0.6.6 A simple graph (i.e., a graph without parallel edges or self-loops) with n vertices and k components can have at most $\dfrac{(n-k)(n-k+1)}{2}$ edges.

Proof. Let the number of vertices in each of the k components of a graph G be $n_1, n_2, n_3, \ldots\ldots, n_k$.
Thus we have $n_1 + n_2 + n_3 + \ldots\ldots + n_k = n$ where $n_i \geq 1$.

The proof of the theorem depends on an algebraic inequality

$\sum_{i=1}^{k}(n_i - 1) = n - k$, squaring both sides ,

$$\left(\sum_{i=1}^{k}(n_i-1)\right)^2 = n^2 + k^2 - 2nk$$

or $\sum_{i=1}^{k}(n_i^2 - 2n_i) + k +$ non-negative cross terms $= n^2 + k^2 - 2nk \because (n_i - 1) \geq 0$, for all i.

$$\therefore \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n = n^2 - (k-1)(2n-k)$$

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$$

Now the maximum number of edges in the i^{th} component of G (which is a simple connected graph) is $\frac{1}{2}n_i(n_i - 1)$. Therefore, the maximum number of edges in G is

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{k} n_i(n_i-1) &= \frac{1}{2}\Big(\sum_{i=1}^{k} n_i^2\Big) - \frac{2}{n} \\ &\leq \frac{1}{2}[n^2-(k-1)(2n-k)] - \frac{2}{n} \\ &= \frac{1}{2}(n-k)(n-k+1)\end{aligned}$$

■

Definition 0.6.21 — EULER GRAPH. If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line and the graph is called an Euler graph

OR

A closed walk running through every edge of the graph G exactly once is called Euler line and a graph having Euler line is called Euler graph.

Theorem 0.6.7 A given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

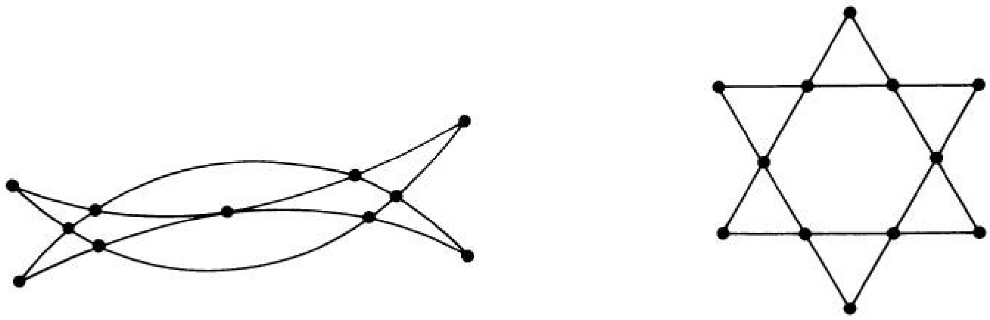
Proof. Suppose that G is an Euler graph therefore it contains an euler line (which is a closed walk). In tracing this walk we observe that every time the walk meets a vertex v it goes through two "new" edges incident on v with one we "entered" v and with the other "exited".This is true not only of all intermediate vertices of the walk but also of the terminal vertex because we "exited" and "entered" the same vertex at the beginning and of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even.
To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edges is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree . we can exit from every vertex we enter; the tracing cannot stop at any vertex but v. And since v is also of even degree, we shall eventually reach v when the tracing comes to an end. If this closed walk h we first traced includes all the edges of G, G is an Euler graph. If not, we remove from G all the edges in h and obtain a subgraph h' of G farmed by the remaining edges. Since both G and h have all their vertices of even degree, the degrees of the vertics of h' are also even. Moreover, h' must touch h at least at one vertex a, because G is connected. Starting from a, we can again construct a new walk in graph h'. Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G . Thus G is an Euler graph. ■

0.7 Konigsberg Bridge Problem:

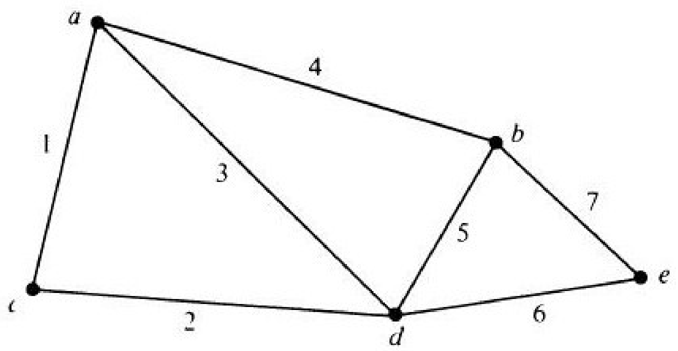
Looking at the graph of the Konigsberg Bridge, we find that not all its vertices are of even degree. Hence, it is not an Euler graph. Thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

■ **Example 0.1** The walk, which includes all the edges of the graph and does not retrace any edges, is not closed. The initial vertex is a and the final vertex is b. We shall call such an open walk that includes (or traces or covers) all edges of a graph without retracing any edge a unicursal graph.
It is clear that by adding an edge between the initial and final vertices of a unicursal line we shall get an Euler line. A connected graph is unicursal if and only if it has exactly two vertices of odd degree. ■

Examples:



Two Euler graphs



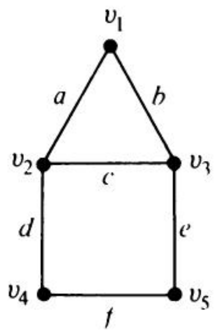
Unicursal graph

Theorem 0.7.1 In a connected graph G with exactly $2k$ odd vertices , there exist k edge- disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

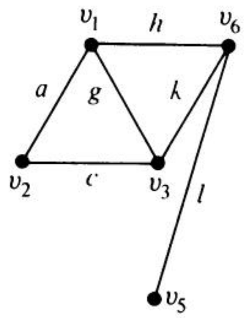
Proof. Let the odd vertices of the given graph G be named $v_1, v_2, ...v_k; w_1, w_2, ...w_k$ in any arbitrary order. Add k edges to G between the vertex pairs $(v_1, w_1), (v_2, w_2), ..., (v_k, w_k)$ to form a new graph G'.
Since every vertex of G' is of even degree, G' consists of an Euler line p. Now if we remove from p the k edges we just added (no two of these edges are incident on the same vertex), p will be split into k walks, each of which is unicursal line: The first removal will leave a single unicursal line; the second removal will split that into unicursal lines; and each successive removal will split a unicursal line into two unicursal lines, until there are k of them. Thus the theorem. ■

Definition 0.7.1 — OPERATIONS ON GRAPHS. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph G_3 (written as $G_3 = G_1 \bigcup G_2$) whose vertex set $V_3 = V_1 \cup V_2$ and the edges $E_3 = E_1 \cup E_2$. The intersection $G_1 \cap G_2$ of graphs G_1 and G_2 is a graph G_4 consisting only of those vertices and edges that are in both G_1 and G_2 . The ring sum of two graphs G_1 and G_2 (written as $G_1 \oplus G_2$) is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2, but not in both.

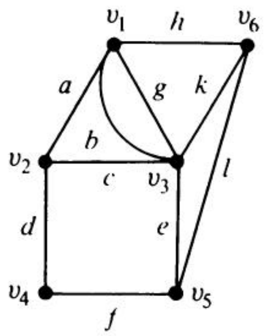
Examples:



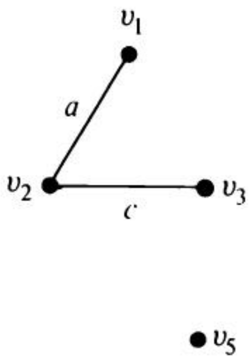
G_1



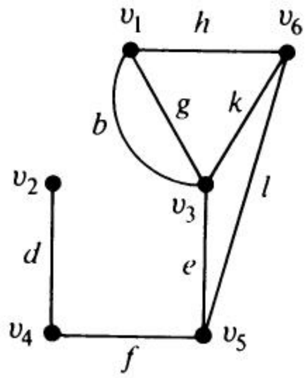
G_2



$G_1 \cup G_2$



$G_1 \cap G_2$



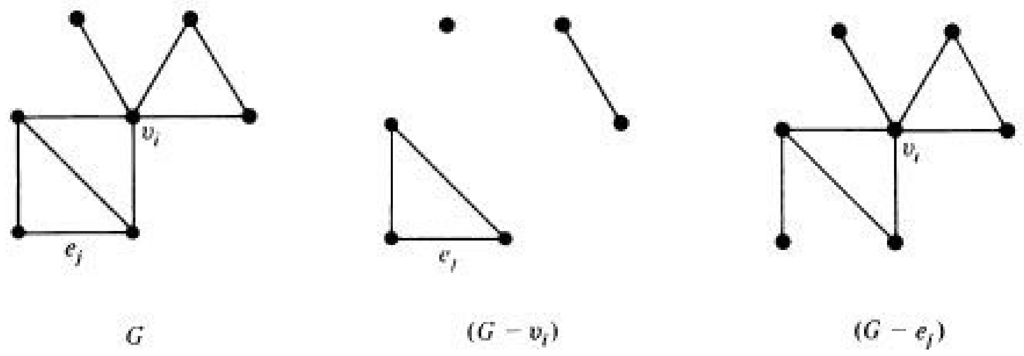
$G_1 \oplus G_2$

Definition 0.7.2 — DECOMPOSITION. A graph G is said to have been decomposed into two subgraphs g_1 and g_2 if $g_1 \cup g_2 = G$ and $g_1 \cap g_2 = a$ null graph.

R A graph containing m edges $\{e_1, e_2, ..., e_m\}$ can be decomposed in $2^{m-1} - 1$ different ways into pair of subgraphs g_1, g_2.

Definition 0.7.3 — DELETION. If v_i is a vertex in graph G, then $G - v_i$ denotes a subgroup of G obtained by deleting (i.e, removing) v_i from G.
Deletion of a vertex always implies the deletion of all edges incident on that vertex.
If e_j is an edge in G, then $G - ej$ is a subgraph of G obtain by deleting ej from G.
Deletion of an edge does not imply deletion of its end vertices.
$\therefore G - e_j = G \oplus e_j$.

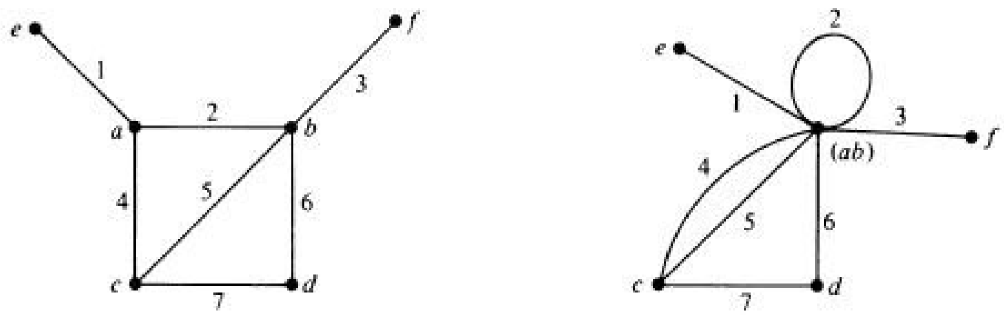
Example:



Vertex deletion and Edge deletion.

Definition 0.7.4 — FUSION. A pair of vertices a, b in a graph are said to be fused (merged) if the two vertices are replacd by a single new vertex such that every edge that was incident on either a or b or on both is incident on the new vertex. Thus fusion of two vertices does not alter the number of edges, but it reduces the number of vertices by one.

Example:



Fusion of vertices a and b.

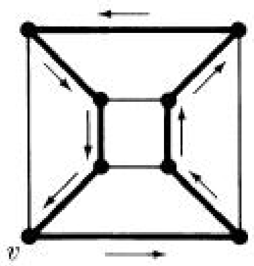
Theorem 0.7.2 A connected graph G is an Euler graph if and only if it can be decomposed into circuits.

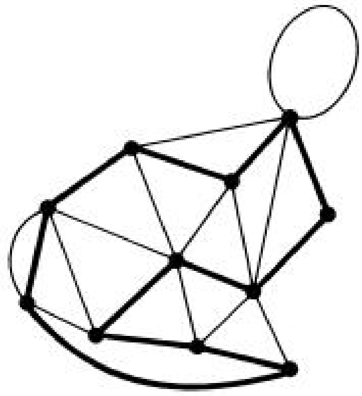
Proof. Suppose graph G can be decomposed into circuits; that is, G is a union of edge- disjoint circuits. Since the degree of every vertex in a circuit is two, the degree of every vertex in G is even. Hence G is an Euler graph.

Conversely, let G be an Euler graph. Consider a vertex v_1. There are at least two edges incident at v_1. Let one of these edges be between v_1 and v_2. Since vertex v_2 is also of even degree, it must have at least another edge, say between v_2 and v_3. Proceeding in this way, we eventually arrive at a vertex that has previously been traversed, thus forming a circuit Γ. Let us remove Γ from G. All vertices in the remaining graph (not necessarily connected) must also be of even degree. From the remaining graph remove another circuit in exactly the same way as we removed Γ from G. Continue this process until no edges are left. Hence the theorem. ■

Definition 0.7.5 — HAMILTONIAN CIRCUIT. A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except at the starting vertex, at which the walk also terminates.

Examples:





Definition 0.7.6 — HAMILTONIAN PATHS. If we remove any one edge from a Hamiltonian circuit, we are left with a path. This path is called a Hamiltonian path.

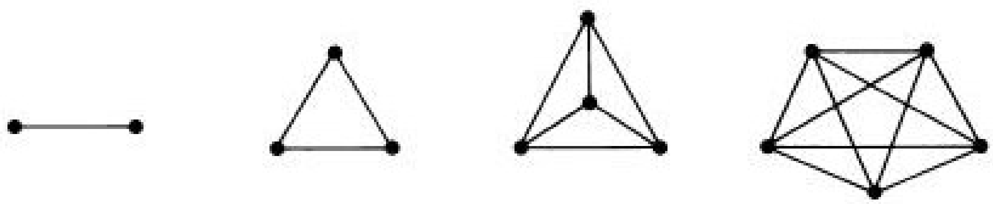
R Hamiltonian path in a graph G traverses every vertex of G.

R Since a Hamiltonian path is a subgraph of a Hamiltonian circuit (which is a subgraph of another graph), every graph that has a Hamiltonian circuit also has a Hamiltonian path.

R The length of a Hamiltonian path (if it exists) in a connected graph of n vertices is $n-1$.

Definition 0.7.7 — COMPLETE GRAPH. A simple graph in which there exists an edge between every pair of vertices is called a complete graph.

Example:



Complete graphs of two, three, four and five vertices.

R A complete graph with n vertices is denoted by K_n.

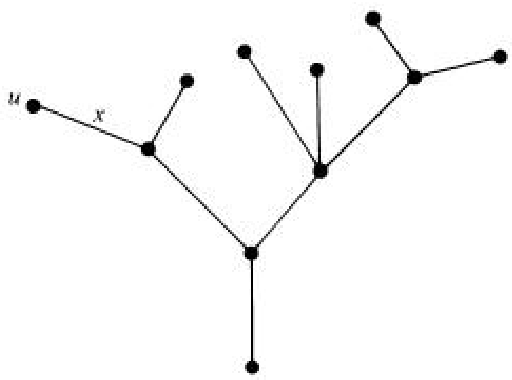
R Since every vertex is joined with every other vertex through one edge, the degree of every vertex is $n-1$ in a complete graph G of n vertices.
$\therefore$ The total number of edges in a complete graph G with n vertices is $\frac{n(n-1)}{2}$.

Definition 0.7.8 — TREE. A tree is a connected graph without any circuits.

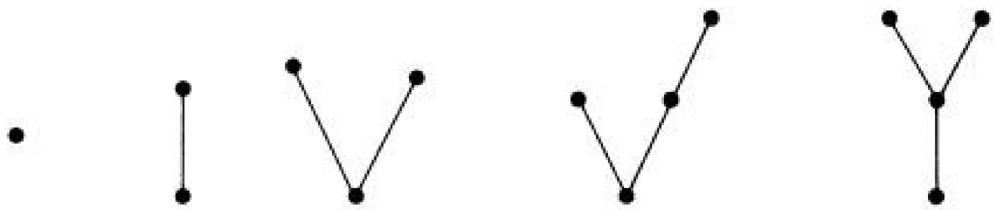
R A tree is a simple graph having neither a self-loop nor parallel edges (because they both form circuits).

R A leaf is a vertex of degree 1.

Examples:



Tree with 11 vertices.



Trees with one, two, three and four vertices.

Theorem 0.7.3 There is one and only one path between every pair of vertices in a tree T.

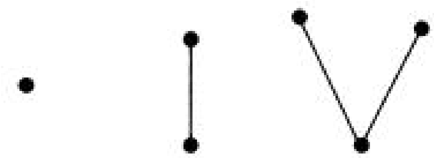
Proof. Since T is a connected graph, therefore there must exist at least one path between every pair of vertices in T. Now suppose that between two vertices a and b of there are two distinct paths. The union of these two paths will contain a circuit and T cannot be a tree. ■

Theorem 0.7.4 If in a graph G there is one and only one path between every pair of vertices, G is a tree.

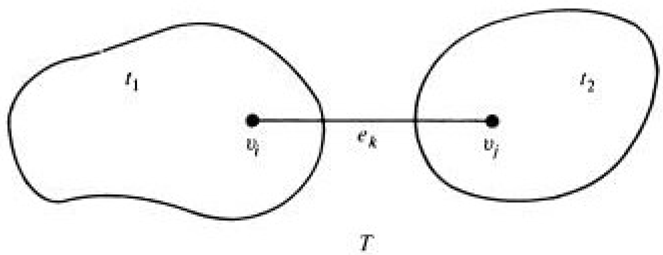
Proof. Existence of a path between every pair of vertices assures that G is connected.
A circuit in a graph (with two or more vertices) implies that there is at least one pair of vertices a, b such that there are two distinct paths between a and b.
Since G has one and only one path between every pair of vertices, G can have no circuit.
$\therefore G$ is a tree. ■

Theorem 0.7.5 A tree with n vertices has $n-1$ edges.

Proof. The theorem will be proved by induction on the number of vertices.
It is easy to see that the theorem is true for $n = 1, 2$ and 3. Assume that the theorem holds for all trees with fewer than n vertices.



Let us now consider a tree T with n vertices. In T let e_k be an edge with end vertices v_i and v_j. According to the well-known Theorem, there is no other path between v_i and v_j except e_k. $\therefore$, Deletion of e_k from T will disconnect the graph, as shown in Figure. Also, $T - e_k$ consists of exactly two components, and since there were no circuits in T to begin with, each of these components is a tree. Both these trees, t_1 and t_2, have fewer than n vertices each, and therefore, by the induction hypothesis, each contains one less edge than the number of vertices in it. Thus $T - e_k$ consists of $n-2$ edges (and n vertices). Hence has exactly $n-1$ edges.



Tree T with n vertices.

■

Theorem 0.7.6 Any connected graph with n vertices and $n-1$ edges is a tree.

Proof. Let G be a connected graph with n vertices and $n-1$ edges. We show that G contains no cycles. Assume to the contrary that G contains cycles. Remove an edge from a cycle so that the resulting graph is again connected. Continue this process of removing one edge from one cycle at a time till the resulting graph H is a tree. As H has n vertices, so number of edges in H is $n-1$. Now, the number of edges in G is greater than the number of edges in H. So $n-1 > n-1$, which is not possible. Hence, G has no cycles and therefore is a tree. ■

Definition 0.7.9 — MINIMALLY CONNECTED GRAPH. A graph is said to be minimally connected if removal of any one edge from it disconnects the graph.

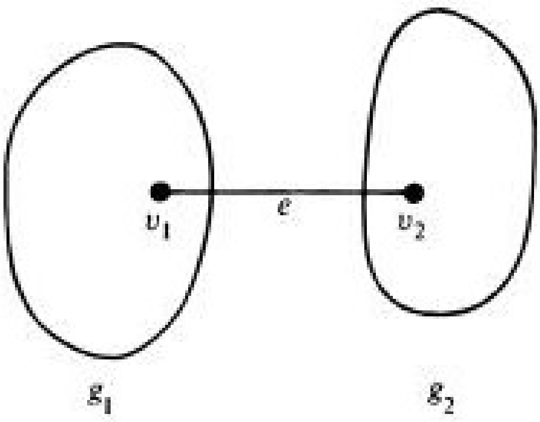
R A minimally connected graph has no cycles.

Theorem 0.7.7 A graph is a tree if and only if it is minimally connected.

Proof. Let the graph G be minimally connected. Then G has no cycles and therefore is a tree. Conversely, let G be a tree. Then G contains no cycles and deletion of any edge from G disconnects the graph. Hence G is minimally connected. ■

Theorem 0.7.8 A graph G with n vertices, $n-1$ edges, and no circuits is connected.

Proof. Suppose there exists a circuitless graph G with n vertices and $n-1$ edges which is disconnected. In that case G will consist of two or more circuitless components. Without loss of generality, let G consist of two components, g_1 and g_2. Add an edge e between a vertex v_1 in g_1 and v_2 in g_2.



Edge added to $G = g_1 \cup g_2$.

Since there was no path between v_1 and v_2 in G, adding e did not create a circuit. Thus $G \cup e$ is a circuitless, connected graph (i.e., a tree) of n vertices and n edges, which is not possible, because we know that A tree with n vertices has $n-1$ edges. ■

R A graph G with n vertices is called a tree if one of the statements is true:

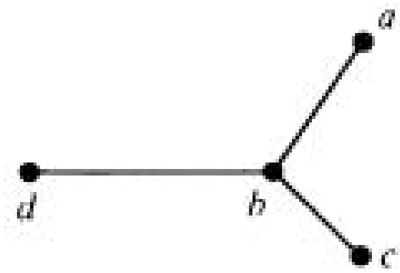
1. G is connected and is circuitless.
2. G is connected and has $n-1$ edges.
3. G is circuitless and has $n-1$ edges.
4. There is exactly one path between every pair of vertices in G.
5. G is a minimally connected graph.

Theorem 0.7.9 In any tree (with two or more vertices), there are at least two pendant vertices.

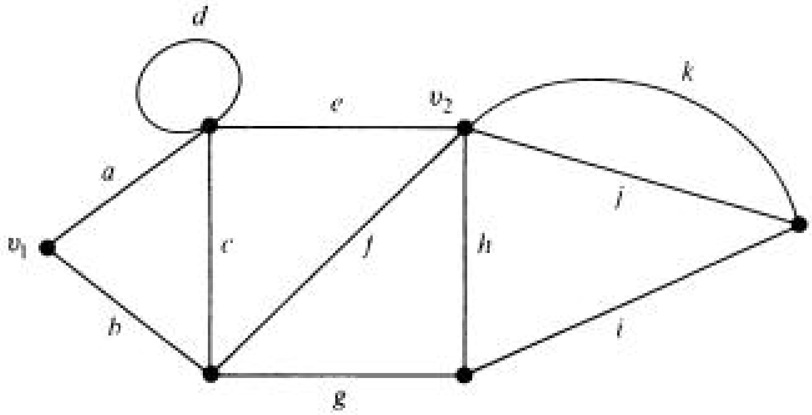
Proof. Let the number of vertices in a given tree T be n (where $n > 1$). So the number of edges in T is $n-1$. Therefore the degree sum of the tree is $2(n-1)$. This degree sum is to be divided among the n vertices. Since a tree is connected it cannot have a vertex of 0 degree. Each vertex contributes at least 1 to the above sum. Thus there must be at least two vertices of degree exactly 1. ■

Definition 0.7.10 — DISTANCE IN A TREE. In a connected graph G, the distance $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path (i.e., the number of edges in the shortest path) between them.

Examples:



$d(a,b) = 1,\ d(a,c) = 2,\ d(a,d) = 2,\ d(b,d) = 1,\ d(c,d) = 2$



Some of the paths between vertices v_1 and v_2 are $(a,e),(a,c,f),(b,c,e),b,f),(b,g,h)$ and (b,g,i,k). There are two shortest paths, (a,e) and (b,f), each of length two. Hence $d(v_1,v_2) = 2$.

Definition 0.7.11 — METRIC. A function $d(u,v)$ is said to be a metric if it satisfies the following three conditions:

1. Non-negativity: $d(u,v) \geq 0$ and $d(u,v) = 0$ iff $u = v$.
2. Symmetry: $d(u,v) = d(v,u)$.
3. Triangle inequality: $d(u,v) \leq d(u,w) + d(w,v)$ for any w.

Theorem 0.7.10 The distance between vertices of a connected graph is a metric.

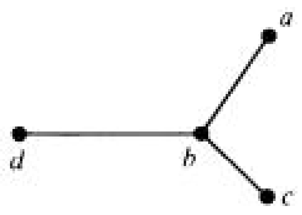
Proof:
Let G be a connected graph and $d(u,v)$ be the distance between vertices of a connected graph, with u and v being vertices of G.

1. Non-negativity: Clearly, $d(u,v) \geq 0$ and $d(u,v) = 0$ iff $u = v$.
2. Symmetry: $d(u,v) = d(v,u)$ because the length of the shortest path from u to v is same as the length of the shortest path from v to u.
3. Triangle inequality: For any vertices u,v,w in G, we have $d(u,v) \leq d(u,w) + d(w,v)$.

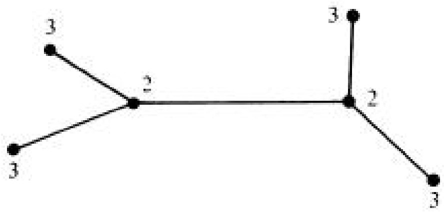
Hence, by definition, $d(u,v)$ is a metric.

Definition 0.7.12 — ECCENTRICITY OF A VERTEX. The eccentricity $E(v)$ of a vertex v in a graph G is the distance from v to the vertex farthest from v in G.

Examples:



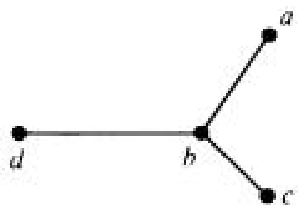
$$E(a) = 2, E(b) = 1, E(c) = 2, E(d) = 2$$



Eccentricities of the vertices of a tree.

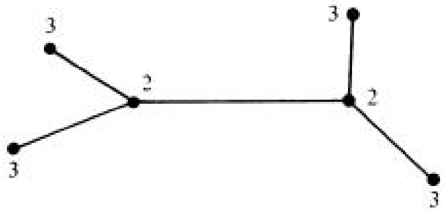
Definition 0.7.13 — CENTER. A vertex with minimum eccentricity in graph G is called a center of G.

Examples:



$$E(a) = 2, E(b) = 1, E(c) = 2, E(d) = 2$$

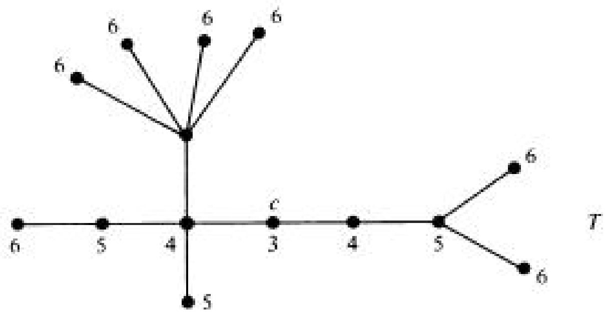
Since minimum eccentricity is 1, therefore center of a graph is b.



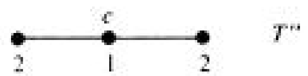
Eccentricities of the vertices of a tree.
Since minimum eccentricity is 2, therefore there are two centers of a graph.

Theorem 0.7.11 Every tree has either one or two centers.

Proof:
The maximum distance, max $d(v, vi)$, from a given vertex to any other vertex v_i occurs only when v_i is a pendant vertex.



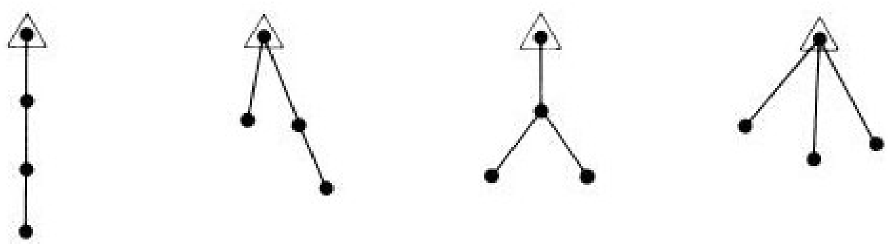
Consider a tree having more than two vertices.
Then Tree must have two or more pendant vertices (By known theorem).
Delete all the pendant vertices from T. The resulting graph T' is still a tree. The removal of all pendant vertices from T uniformly reduces the eccentricities of the remaining vertices (i.e., vertices in T') by one. Therefore, all vertices that had as centers will still remain centers in T'. From T' we can again remove all pendant vertices and get another tree T''. Continue this process, until there is left either a vertex (which is the center of T) or an edge (whose end vertices are the two centers of T). Thus the theorem.



R If a tree T has two centers, then the two centers must be adjacent.

Definition 0.7.14 — ROOTED TREE. A tree in which one vertex (called the root) is distinguished from all the others is called a rooted tree.

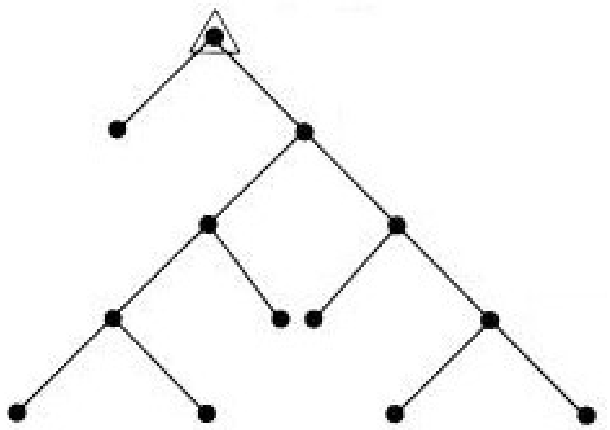
Examples:

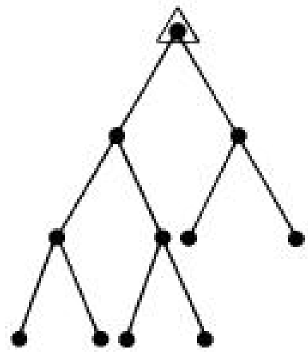


Rooted tree with four vertices.

Definition 0.7.15 — BINARY TREE. A binary tree is defined as a tree in which there is exactly one vertex of degree two and each of the remaining vertices is of degree one or three.

Examples:





R Every binary tree is a rooted tree.

R A non-pendant vertex in a tree is called an internal vertex.

R The number of internal vertices in a binary tree is one less than the number of pendant vertices.

R

Let p be the number of pendant vertices in a binary tree T. Then $n-p-1$ is the number of vertices of degree three.
We know that the sum of the degrees of all vertices in T is twice the number of edges in T. That is,

$$\begin{aligned} & p+3(n-p-1)+2=2(n-1) \\ \Longrightarrow\ & p+3n-3p-3+2=2n-2 \\ \Longrightarrow\ & 2p=n+1 \\ \Longrightarrow\ & p=\frac{n+1}{2} \end{aligned}$$

Corollary 0.7.12 The number of vertices in a binary tree is always odd.

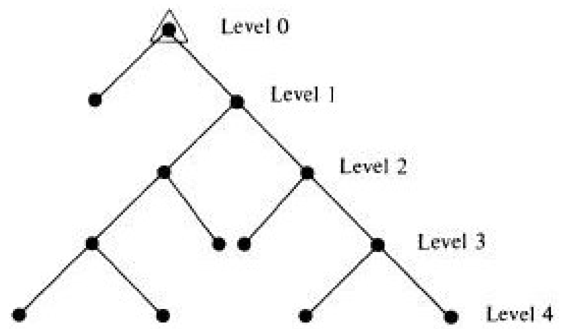
Proof. Let the number of vertices in a binary tree be n.
In a binary tree, there is exactly one vertex of even degree, and the remaining $n-1$ vertices are of odd degrees.
We know that the number of vertices of odd degrees is even, therefore $n-1$ is even.
Hence n is odd. ■

Definition 0.7.16 — LEVEL OF A VERTEX. In a binary tree a vertex v_i is said to be at level l_i if v_i is at a distance of l_i from the root.

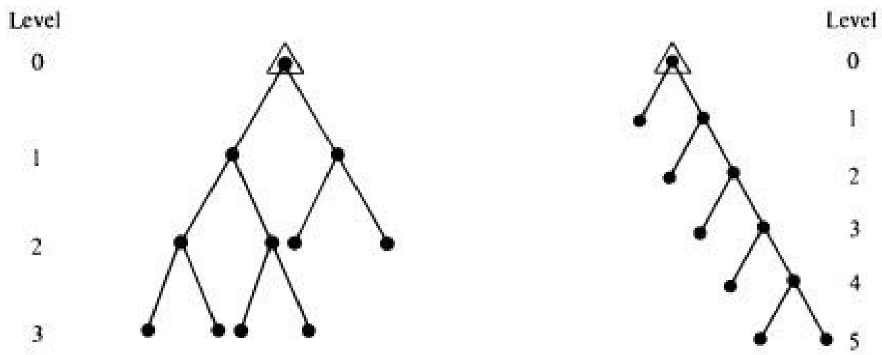
R The root is at level 0.

R The maximum level ($l_{\max}$), of any vertex in a binary tree is called the height of the tree.

Examples:



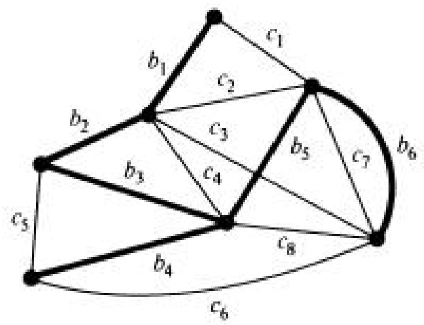
A 13-vertex, 4-level binary tree.



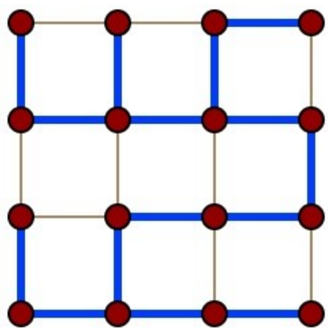
11-vertex binary trees.

Definition 0.7.17 — SPANNING TREE. A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G.

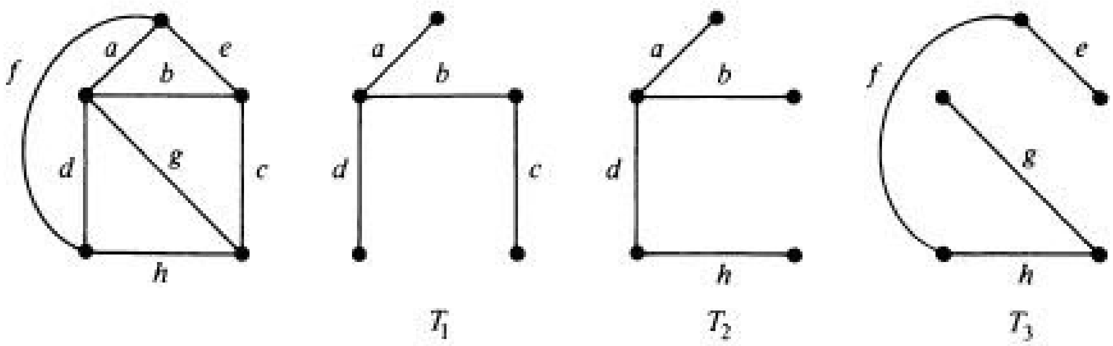
R An edge in a spanning tree T is called a **branch** of the spanning tree T.
If an edge of a connected graph G is not a branch of a spanning tree T, then the edge is called a **chord** of the spanning tree T.
Examples:



The edges $b_1, b_2, \ldots, b_6$ are branches of the tree indicated by thick edges.



Thick line is the spanning tree.



Graph and three of its spanning trees.

Theorem 0.7.13 Every connected graph has at least one spanning tree.

Proof. Let G be a connected graph.
If G has no circuit, then G is a spanning tree.
If G has a circuit, then delete an edge from this circuit, till the graph is connected.
If there are more circuits, repeat the process till an edge from the last circuit is deleted, leaving the graph connected, circuitless and contains all the vertices of G.
Thus the subgraph obtained is a spanning tree of G.
Hence every connected graph has at least one spanning tree. ■

Theorem 0.7.14 With respect to any of its spanning trees, a connected graph of n vertices and e edges has $n-1$ tree branches and $e-n+1$ chords.

Proof. Let G be a connected graph with n vertices and e edges.
Let T be any spanning tree in G.
Since every spanning tree of G contains all the vertices of G.
$\therefore$ Total number of vertices in T is n and total number of edges in T is $n-1$.
Since every edge of a spanning tree T is called a branch of T, therefore G contains $n-1$ branches w.r.to T.
Since the number of edges in G is e, therefore the number of chords in G w.r.to T is

$$e-(n-1)=e-n+1$$

. ■

Definition 0.7.18 — RANK AND NULLITY OF A GRAPH. If a graph G has n vertices, e edges and k components, then

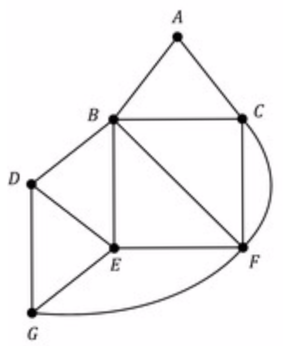
$$\boxed{\text{Rank is } r=n-k \text{ and Nullity is } \mu=e-n+k}$$

R Rank of G is the number of branches in any spanning tree of G.
Nullity of G is the number of chords in G.
Rank+Nullity= number of edges in G.

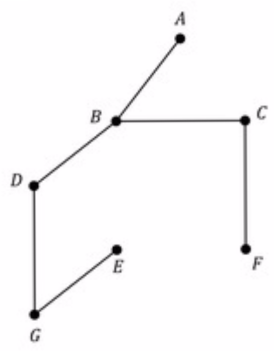
R If G is a connected graph, then rank $=n-1$ and nullity$=e-n+1$.

Definition 0.7.19 — FUNDAMENTAL CIRCUIT. Consider a spanning tree T in a connected graph G. Adding any one chord to T will create exactly one circuit. Such a circuit, formed by adding a chord to a spanning tree, is called a fundamental circuit.

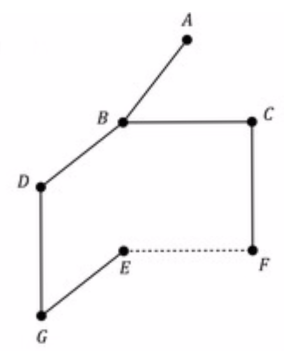
Examples:



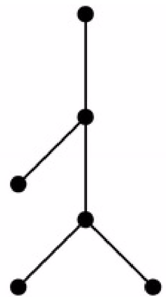
Graph



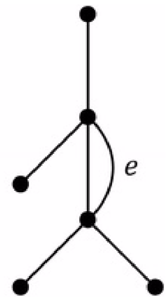
Spanning tree



EGDBCFE - Fundamental circuit



Tree

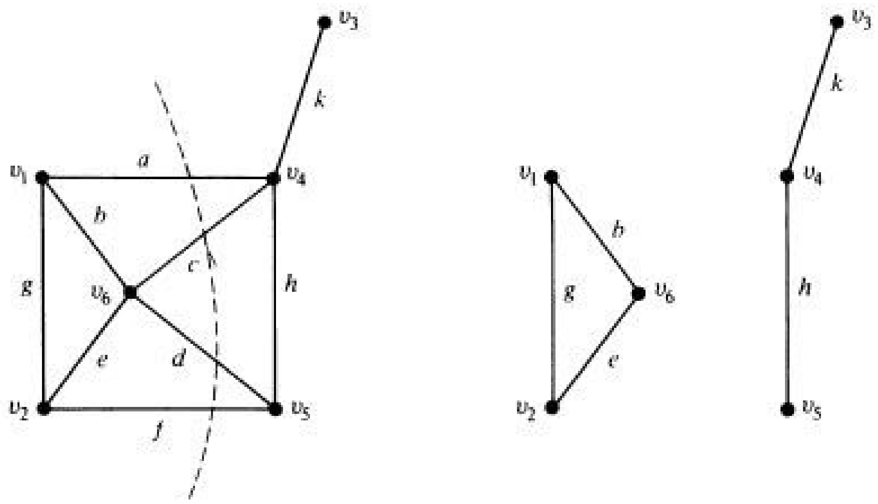


Fundamental circuit

Unit 2

> **Definition 0.7.20 — CUT-SET.** In a connected graph G, a cut-set is a set of edges whose removal from graph G leaves G disconnected, provided removal of no proper subset of these edges disconnects G.

Example:



Removal of a cut-set $\{a,c,d,f\}$ from a graph cuts it into two.
In figure, the cut-sets are: $\{a,c,d,f\},\{a,b,g\},\{a,b,e,f\},\{d,h,f\},\{k\}$.

R Rank of G is the number of branches in any spanning tree of G.
Nullity of G is the number of chords in G.
Rank+Nullity= number of edges in G.

R **Every edge of a tree is a cut-set** because removal of any edge from a tree cuts the tree into two parts.

Theorem 0.7.15 Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.

Proof. Let G be a connected graph and S be a cut-set of G.
If possible, suppose T is spannig tree of G which has no edge included in the cut-set S. Therefore T is completely contained in $G-S$. As T is a spanning tree and spans through all the vertices of G, the subgraph $G-S$ remains connected.
But, that is not possible as removal of a cut-set must leave the graph disconnected.
Therefore, our assumption is wrong. Hence, every cut-set in a connected graph G must contain at least one branch of every spanning tree.

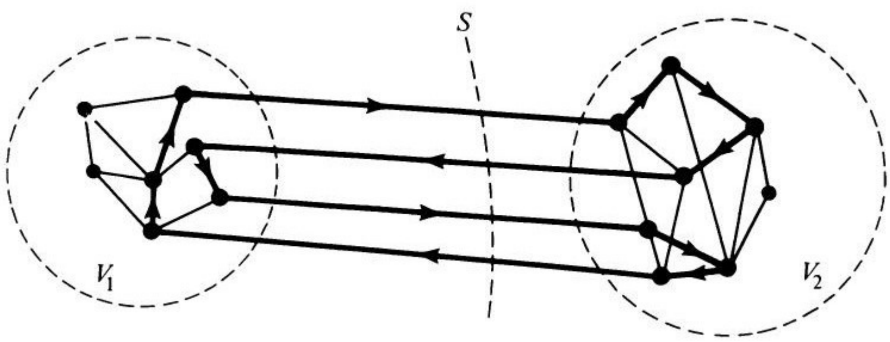
■

Theorem 0.7.16 In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cut-set.

Proof. Let G be a connected graph and Q be a minimal set of edges containing at least one branch of every spanning tree of G.
Now, $G-Q$ is a subgraph of G from which at least one branch of every spanning tree is missing.
As $G-Q$ cannot contain any spanning tree of G completely, it must be disconnected. Since, Q is a minimal set of edges with this property, any edge e returned from G to $G-Q$ will create at least one spanning tree. Therefore, $G-Q+e$ will be a connected graph.
Thus, Q is a minimal set of edges whose removal from G disconnects G.
Hence, Q is a cut-set of G. ■

Theorem 0.7.17 Every circuit has an even number of edges in common with any cut-set.

Proof. Consider a cut-set S in graph G. Let the removal of S partition the vertices of G into two (mutually exclusive or disjoint) subsets V_1 and V_2. Consider a circuit Γ in G. If all the vertices in Γ are entirely within vertex set V_1 (or V_2), the number of edges common to S and Γ is zero; that is, $N(S \cap T) = 0$, an even number; where $N(S \cap T)$ is the number of edges in subgraph $S \cap T$.

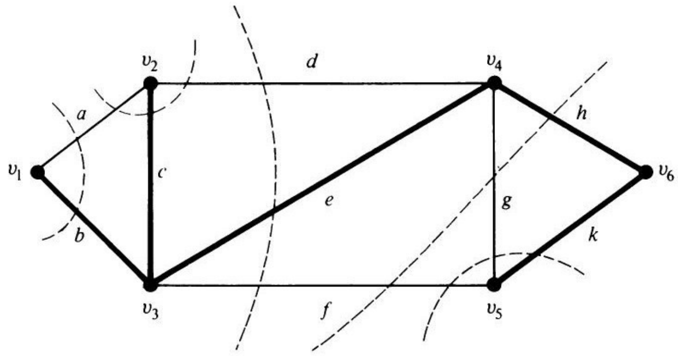


Circuit Γ shown in heavy lines and is traversed along the direction of the arrows.
Circuit and a cut-set in G.

On the other hand, if some vertices in Γ are in V_1 and some in V_2, we traverse back and forth between the sets V_1 and V_2 as we traverse the circuit. Because of the closed nature of a circuit, the number of edges we traverse between V_1 and V_2 must be even. And since very edge in S has one end in V_1 and the other in V_2, and no other edge in G has this property (of separating sets V_1 and V_2), the number of edges common to S and Γ is even. ■

Definition 0.7.21 — FUNDAMENTAL CUT-SET. Consider a spanning tree T of a connected graph G. Take any branch b in T. Since b is a cut-set in T, $\{b\}$ partitions all vertices of T into two disjoint sets—one at each end of b. Consider the same partition of vertices in G, and the cut set S in G that corresponds to this partition. Cutset S will contain only one branch b of T, and the rest (if any) of the edges in S are chords with respect to T. Such a cut-set S containing exactly one branch of a tree T is called a fundamental cut-set with respect to T.

Example:



Fundamental cut-sets of a graph.

Theorem 0.7.18 The ring sum of any two cut-sets in a graph is either a third cut-set or an edge-disjoint union of cut-sets.

Theorem 0.7.19 With respect to a given spanning tree T, a chord c_i that determines a fundamental circuit Γ occurs in every fundamental cut-set associated with the branches in Γ and in no other.

Proof. Let Γ be the fundamental circuit of a connected graph G w.r.to a chord c_i of a spanning tree T. Let Γ consist of k branches $b_1, b_2, b_3, \ldots, b_k$ in addition to the chord c_i.
that is, $\Gamma = \{c_i, b_1, b_2, b_3, \ldots, b_k\}$ is a fundamental circuit with respect to T. Every branch of any spanning tree has a fundamental cut-set associated with it.
Let S_1 be the fundamental cut-set associated with b_1 consisting of q chords in addition to the branch b_1; that is, $S_1 = \{b_1, c_1, c_2, c_3, \ldots, c_q\}$ is a fundamental cut-set w.r.to T.
We know that "Every circuit has an even number of edges in common with any cut-set", there must be an even number of edges common to Γ and S_1. Edge b_1 is in both Γ and S_1 and there is only one other edge in Γ (which is c_i) that can possibly also be in S_1. Therefore, we must have two edges b_1 and c_i common to S_1 and Γ. Thus the chord c_i is one of the chords Exactly the same argument holds for fundamental cut-sets associated with $b_2, b_3, \ldots, b_k$. Therefore, the chord c_i is contained in every fundamental cut-set associated with branches in Γ.

■

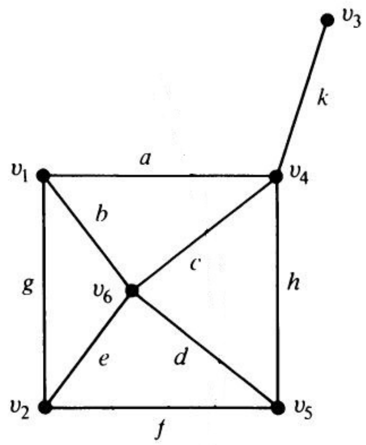
Theorem 0.7.20 With respect to a given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S, and in no others.

Proof. Let the fundamental cut-set determined by a branch b_i of spanning tree T be $S = \{b_i, c_1, c_2, c_3, ..., c_p\}$, where $c_1, c_2, c_3, ..., c_p$ are chords of T.
Let $\Gamma_1 = \{c_i, b_1, b_2, b_3, ..., b_q\}$, where $b_1, b_2, b_3, ..., b_q$ are branches of T.
We know that "Every circuit has an even number of edges in common with any cut-set", the number of edges common to S and Γ_1 must be even. Therefore, b_i must be in Γ_1. Exactly the same argument holds for the fundamental circuits associated with chords $c_2, c_3, \ldots, c_p$.
Now, we show that no other fundamental circuit of T contains b_i (besides those associated with $c_1, c_2, c_3, \ldots, c_p$). Suppose that b_i occurs in a fundamental circuit Γ_{p+1} made by a chord other than $c_1, c_2, c_3, \ldots, c_p$.
Then b_i is the only edge common to Γ_{p+1} and S, which is not possible. Hence the theorem. ■

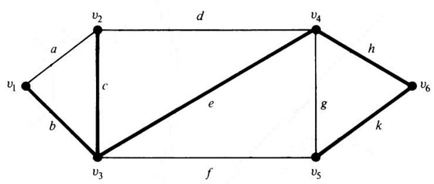
Definition 0.7.22 — VERTEX CONNECTIVITY. The vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from graph G leaves the remaining graph disconnected.

R The vertex connectivity of a tree is 1.

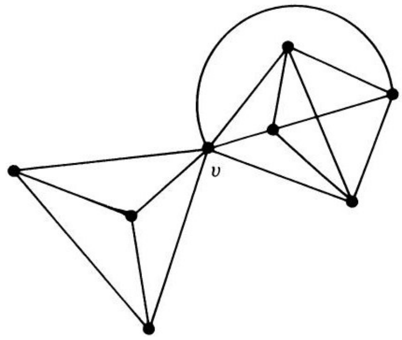
Examples:



Vertex Connectivity is 1



Vertex Connectivity is 2

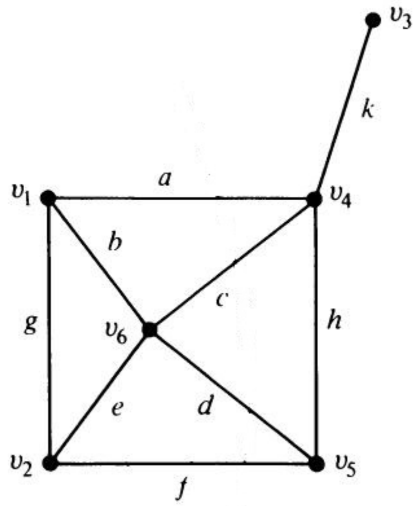


Vertex Connectivity is 1

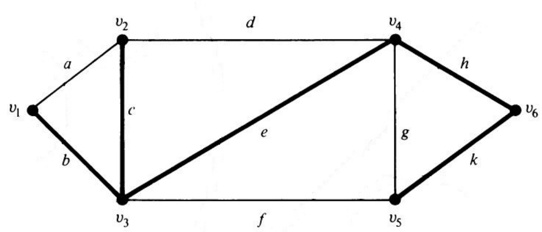
> **Definition 0.7.23 — EDGE CONNECTIVITY.** The edge connectivity of a connected graph G is defined as the minimum number of edges whose removal reduces the rank of the graph by one.

R The edge connectivity of a tree is 1.

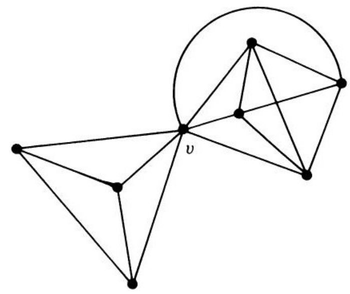
Examples:



Edge Connectivity is 1



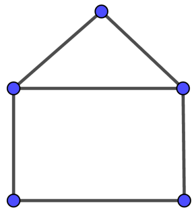
Edge Connectivity is 2



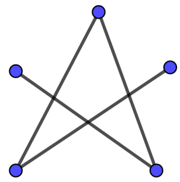
Edge Connectivity is 3

> **Definition 0.7.24 — COMPLEMENT.** Complement of a simple graph G is a simple graph $\overline{G}$ having the same vertices of G and the vertices which are adjacent in G are not adjacent in $\overline{G}$.

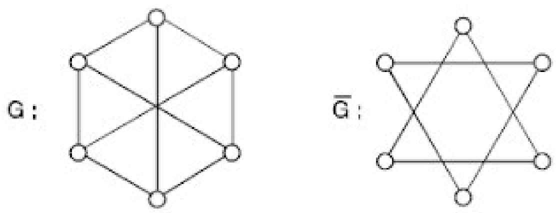
Examples:

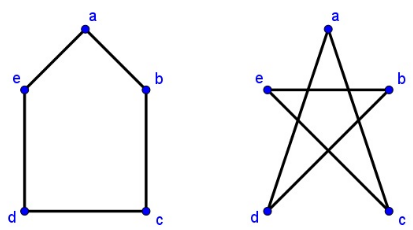


G



$\overline{G}$





G $\overline{G}$

R Number of vertices in G = Number of vertices in $\overline{G}$.

R No. of edges in G+ No. of edges in $\overline{G}$ = No. of edges in a complete graph $= \dfrac{n(n-1)}{2}$.

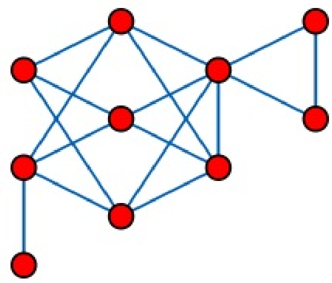
R Complement of a complete graph is always a null graph.

Definition 0.7.25 — SEPARABLE GRAPH. A connected graph is said to be separable if its vertex connectivity is one. All other connected graphs are called non-separable.

OR

A connected graph G is said to be separable if there exists a subgraph g in G such that $\overline{g}$ (the complement of g in G) and g have only one vertex in common.

Example:



R In a separable graph a vertex whose removal disconnects the graph is called a cut-vertex or a cut-node or an articulation point.

Theorem 0.7.21 A vertex v in a connected graph G is a cut-vertex if and only if there exist two vertices x and y in G such that every path between x and y passes through v.

Theorem 0.7.22 The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

Proof. Let vertex v_i be the vertex with the smallest degree in G. Let $d(v_i)$ be the degree of v_i. Vertex v_i can be separated from G by removing the $d(v_i)$ edges incident on vertex v_i. Hence the theorem. ∎

Theorem 0.7.23 The vertex connectivity of any graph G can never exceed the edge connectivity of G.

Proof. Let α denote the edge connectivity of G. Therefore, there exists a cutset S in G with α edges. Let S partition the vertices of G into subsets V_1 and V_2. By removing at most α vertices from V_1 (or V_2) on which the edges in S are incident, we can effect the removal of S (together with all other edges incident on these vertices) from G. Hence the theorem. ∎

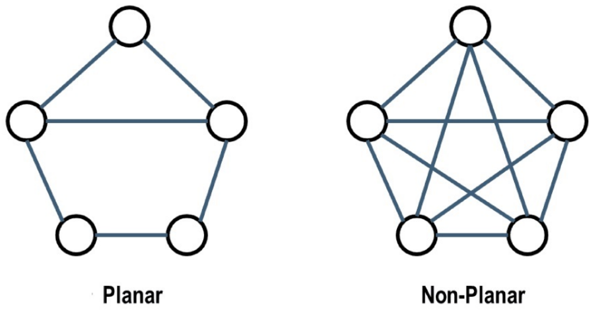
R Every cut-set in a non-separable graph with more than two vertices contains at least two edges.

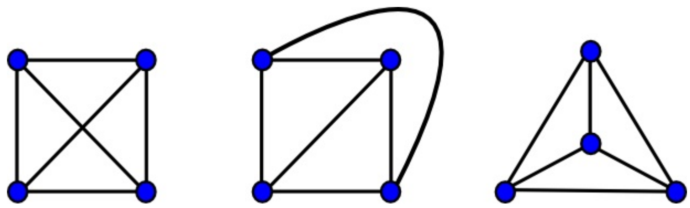
Theorem 0.7.24 The maximum vertex connectivity one can achieve with a graph G of n vertices and e edges $(e \geq n-1)$ is the integral part of the number $\frac{2e}{n}$; that is, $\left[\frac{2e}{n}\right]$.

Definition 0.7.26 — PLANAR AND NON-PLANAR GRAPHS. A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.
A graph that cannot be drawn on a plane without a crossover between its edges is called non-planar.

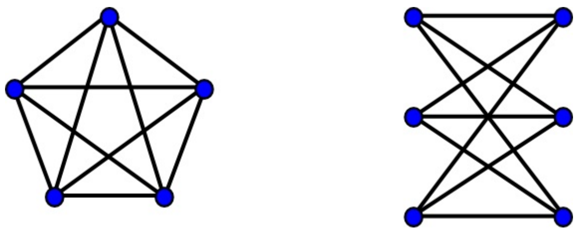
R A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding.

Examples:





Planar graphs (Same graph can be drawn in different ways).



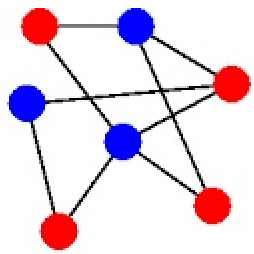
Two non-planar graphs.

Definition 0.7.27 — BIPARTITE GRAPH. A graph G is called bipartite if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2.

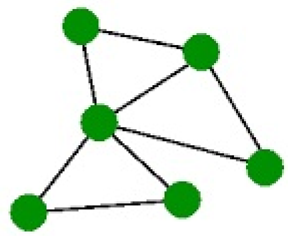
R Every tree is a bipartite graph.

R Bipartite graph is denoted by $K_{m,n}$, where m and n are the numbers of vertices in V_1 and V_2 respectively.

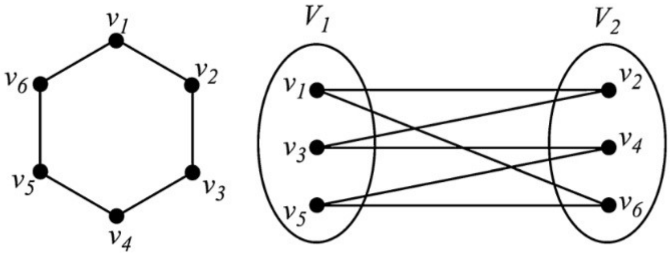
Examples:



Bipartite Graph



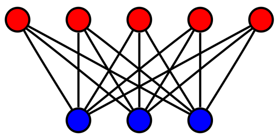
non–Bipartite Graph



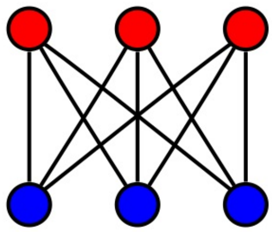
Bipartite graph.

> **Definition 0.7.28 — COMPLETE BIPARTITE GRAPH.** A bipartite graph is said to be a complete bipartite graph if there is one edge between every vertex of set V_1 to every vertex of set V_2.

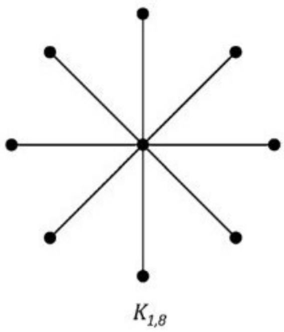
Examples:



Complete bipartite graph $-$ $K_{5,3}$.



Complete bipartite graph $-$ $K_{3,3}$.



Complete bipartite graph $K_{1,8}$.

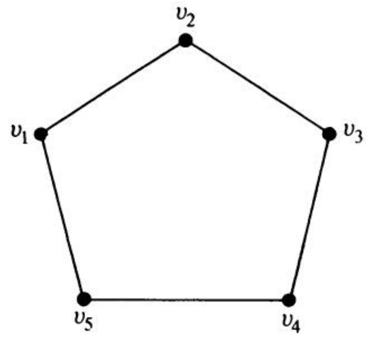
0.8 KURATOWSKI'S TWO GRAPHS

Theorem 0.8.1 The complete graph of five vertices (K_5) is non-planar.

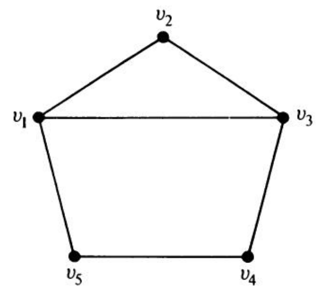
OR

Kuratowski's first graph is non-planar.

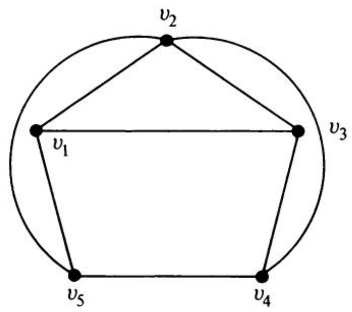
Proof. Let the five vertices in the complete graph (K_5) be named v_1, v_2, v_3, v_4 and v_5. A complete graph is a simple graph in which every vertex is joined to every other vertex by means of an edge. So, we must have a circuit going from v_1 to v_2 to v_3 to v_4 to v_5 to v_1 - that is, a pentagon. This pentagon divides the plane into two regions, one inside and the other outside.



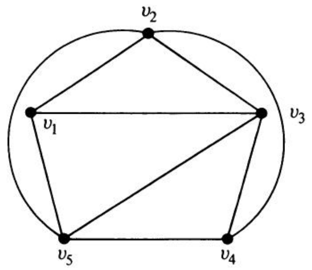
Since vertex v_1 is to be connected to v_3 by means of an edge, this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously). Suppose that we choose to draw a line from v_1 to v_3 inside the pentagon.



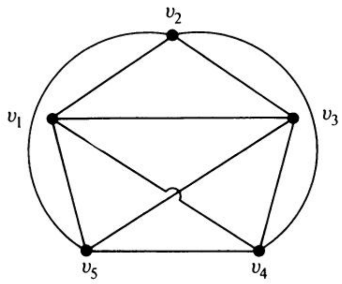
Now we have to draw an edge from v_2 to v_4 and another one from v_2 to v_5. Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pentagon.



The edge connecting v_3 and v_5 cannot be drawn outside the pentagon without crossing the edge between v_2 and v_4. Therefore, v_3 and v_5 have to be connected with an edge inside the pentagon.



Now we have yet to draw an edge between v_1 and v_4. This edge cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane. Hence the graph K_5 is non-planar.



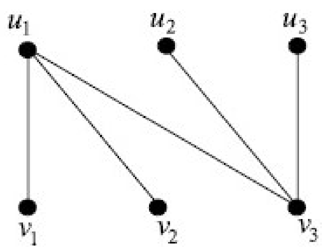
■

Theorem 0.8.2 The complete bipartite graph $(K_{3,3})$ is nonplanar.

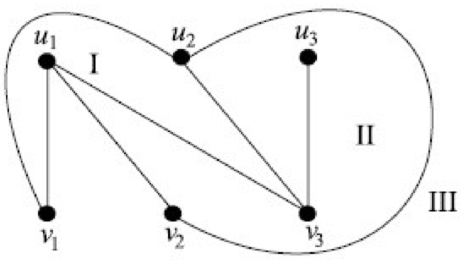
OR

Kuratowski's second graph is non-planar.

Proof. The complete bipartite graph has six vertices and nine edges. Let the vertices be $u_1, u_2, u_3, v_1, v_2, v_3$. We have edges from every u_i to each v_i, where $1 \leq i \leq 3$. First we take the edges from u_1 to each v_1, v_2 and v_3.



Then we take the edges between u_2 to each v_1, v_2 and v_3. Thus we get three regions namely I, II and III.



Finally we have to draw the edges between u_3 to each v_1, v_2 and v_3. We can draw the edges between u_3 and v_3 inside the region II without any crossover. We can also draw the edges between u_3 and v_2 inside the region II without any crossover. But the edge from u_3 to v_1 can not be drawn in any region without having a crossover with the previous edges. Thus the graph cannot be embedded in a plane. Hence $K_{3,3}$ is nonplanar. ■

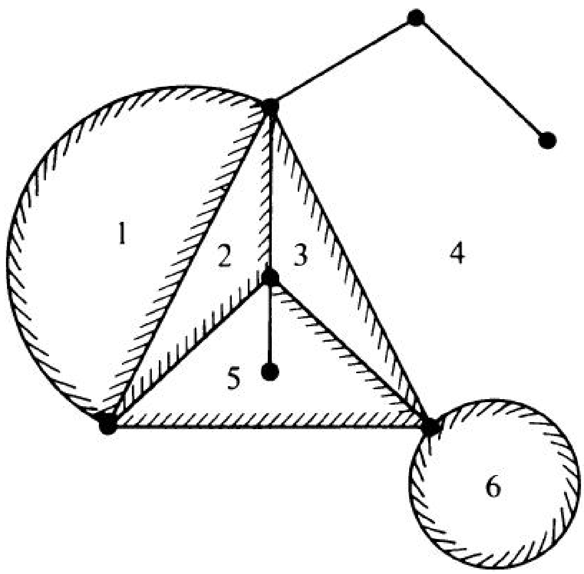
R

1. Both K_5 and $K_{3,3}$ are regular graphs.
2. Both K_5 and $K_{3,3}$ are non-planar graphs.
3. In K_5 and $K_{3,3}$, Removal of one edge or a vertex makes each a planar graph.
4. Kuratowski's first graph is the non-planar graph with the smallest number of vertices and Kuratowski's second graph is the non-planar graph with the smallest number of edges.
5. Both K_5 and $K_{3,3}$ are the simplest non-planar graphs.

Theorem 0.8.3 Any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

Definition 0.8.1 — REGION. A plane representation of a graph divides the plane into regions (or faces).

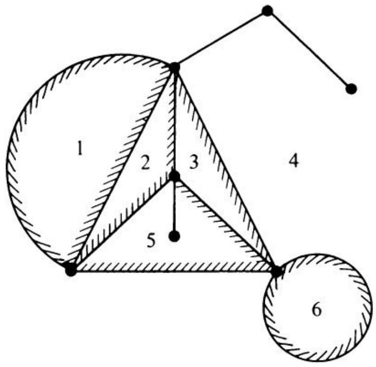
Example:



Plane representation with six regions.

Definition 0.8.2 — INFINITE REGION. The portion of the plane lying outside a graph embedded in a plane is called the infinite (or unbounded or outer or exterior) region.

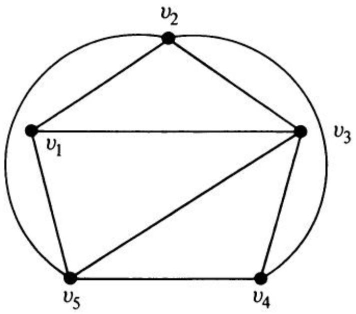
Example:



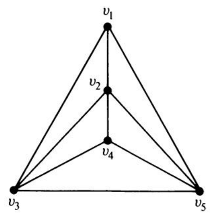
Region 4 is an infinite region.

R Any simple planar graph can be drawn without crossings so that its edges are straight line segments.

Example:



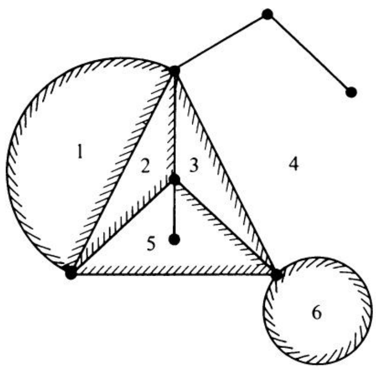
Graph G



Straight-line representation of the graph G

Theorem 0.8.4 A connected planar graph with n vertices and e edges has $e-n+2$ regions.

Proof. It is sufficient to prove for a simple graph, because adding a self-loop or a parallel edge just adds one region to the graph and simultaneously increases the value of e by one. We can also remove all edges that do not form boundaries of any region. Three such edges are shown in the following figure:



Adding (or removing) of any such edge increases (or decreases) e by one and increases (or decreases) n by one, keeping the value $e-n$ unchanged.
Since any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon. Let the polygon representing the given graph consist of f regions and let k_p be the number of p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3\cdot k_3+4\cdot k_4+5\cdot k_5+\ldots\ldots\ldots\ldots..+r\cdot k_r=2e \tag{1}$$

where k_r is the number of polygons with maximum edges.
Also,

$$k_3+k_4+k_5+\ldots\ldots\ldots\ldots..+k_r=f \tag{2}$$

$$\text{The sum of all angles subtended at each vertex in the polygon} = 2\pi n \tag{3}$$

We know that the sum of all interior angles of a p-sided polygon is $\pi(p-2)$ and the sum of the exterior angles is $\pi(p+2)$, now finding the expression in (3) as
the grand total of all interior angles of $f-1$ finite regions + the sum of the exterior angles of the polygon = the infinite region. i.e.,

$$\pi(\overset{\checkmark}{3}-\overset{\times}{2})\cdot k_3+\pi(\overset{\checkmark}{4}-\overset{\times}{2})\cdot k_4+\pi(\overset{\checkmark}{5}-\overset{\times}{2})\cdot k_5+\ldots\ldots\ldots\ldots..+\pi(\overset{\checkmark}{r}-\overset{\times}{2})\cdot k_r+4\pi=2\pi n$$

$$\pi\{3\cdot k_3+4\cdot k_4+5\cdot k_5+\ldots\ldots\ldots\ldots..+r\cdot k_r\}-2\pi\{k_3+k_4+k_5+\ldots\ldots\ldots\ldots..+k_r\}+4\pi=2\pi n$$

$$\pi\cdot 2e-2\pi\cdot f+4\pi=2\pi n \qquad \because \text{from (1) and (2)}$$

Dividing throughout by 2π, we get

$$e-f+2=n$$

Hence, the number of regions is given by

$$\boxed{f=e-n+2}$$

(Euler's formula) ■

Corollary 0.8.5 In any simple connected planar graph with f regions, n vertices and e edges $(e>2)$, the following inequalities must hold:

$$e\geq\frac{3}{2}f$$

$$e\leq 3n-6$$

Proof. Since each region is bounded by at least three edges and each edge belongs to exactly two regions,

$$2e \geq 3f \implies e \geq \frac{3}{2}f \tag{4}$$

Substituting for f from Euler's formula in inequality (4), we get

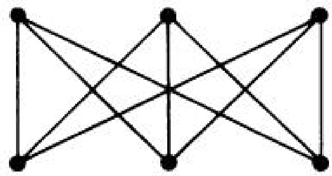
$$e \geq \frac{3}{2}(e-n+2) \implies 2e \geq 3e-3n+6 \implies 3n-6 \geq e \implies e \leq 3n-6$$

■

R The inequality $e \leq 3n-6$ is useful in finding out if a graph is non-planar.

■ **Example 0.2** Prove that a complete graph with 5 vertices K_5 is non-planar.
Solution: $n = 5, e = 10$
Consider, $3n-6 = 3 \cdot 5 - 6 = 9 < e$ but $e \leq 3n-6$.
$\implies$ The graph K_5 is non-planar. ■

■ **Example 0.3** Prove that $K_{3,3}$ is non-planar.
Solution: In $K_{3,3}$, $n = 6, e = 9$.



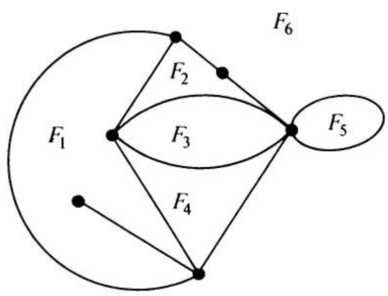
Suppose $K_{3,3}$ is planar. Then in this planar representation of graph, there is no region less than 4 edges and so

$$\begin{aligned} 2e &\geq 4f \\ 2e &\geq 4(e-n+2) \quad \text{(by Euler's formula } f = e-n+2) \\ 2 \times 9 &\geq 4(9-6+2) \\ 18 &\geq 20, \quad \text{is a contradiction} \end{aligned}$$

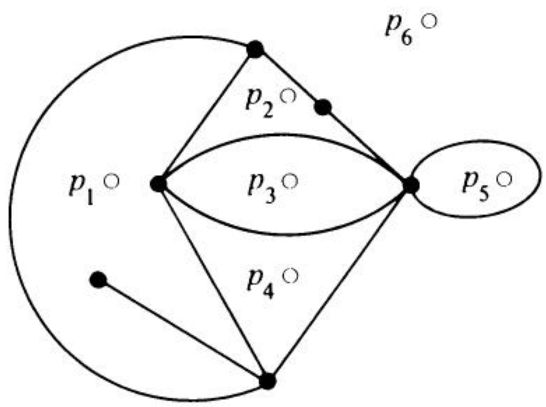
Hence, a graph $K_{3,3}$ is non-planar. ■

0.9 Geometric Dual:

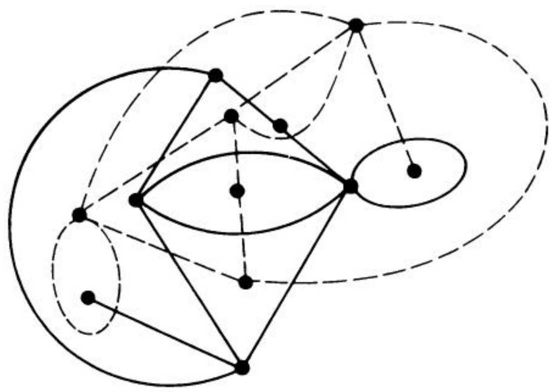
Consider the plane representation of a graph in figure, with six regions or faces F_1, F_2, F_3, F_4, F_5 and F_6.



Now place six points p_1, p_2, p_3, p_4, p_5 and p_6, one in each of the regions, as shown in figure:



Next join these six points according to the following procedure:
If two regions F_i and F_j are adjacent (i.e., having a common edge), draw a line joining points p_i and p_j that intersects the common edge between F_i and F_j exactly once. If there is more than one edge common between F_i and F_j, draw one line between points p_i and p_j for each of the common edges. For an edge e lying entirely in one region, say F_k, draw a self-loop at point p_k intersecting e exactly once.
By this procedure we get a new graph G^* [in dotted lines] consisting of six vertices p_1, p_2, p_3, p_4, p_5 and p_6 and of edges joining these vertices. Such a graph G^* is called a dual of G.

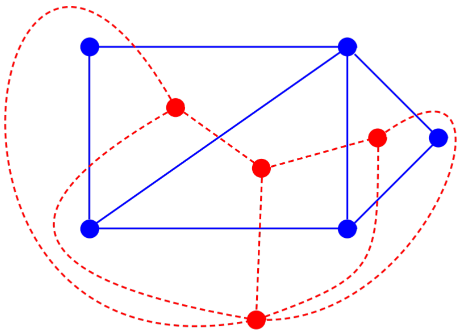


Relationship between a planar graph G and its dual G^* are:

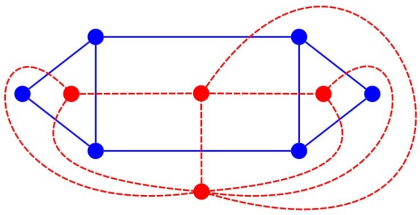
1. An edge forming a self-loop in G yields a pendant edge in G^*.
2. A pendant edge in G yields a self-loop in G^*.
3. Edges that are in series in G produce parallel edges in G^*.
4. Parallel edges in G produce edges in series in G^*.
5. Graph G^* is also embedded in the plane and is therefore planar.
6. Considering the process of drawing a dual G^* from G, it is clear that G is a dual of G^*. Therefore, G and G^* are dual graphs.
7. If n, e, f, r and μ denote the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and if n^*, e^*, f^*, r^* and μ^* are the corresponding numbers in dual graph G*, then

$$\left.\begin{aligned} n^* &= f \\ e^* &= e \\ f^* &= n \end{aligned}\right\} \implies r^* = \mu,\ \mu^* = r.$$

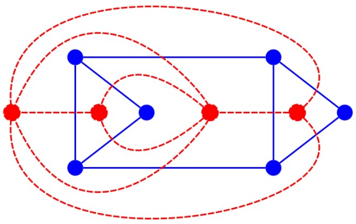
Examples:



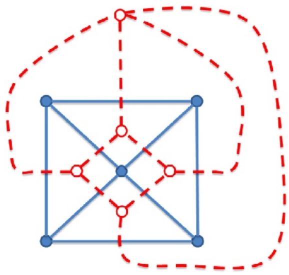
The red graph is the dual graph of the blue graph and vice versa.



The red graph is the dual graph of the blue graph and vice versa.



The red graph is the dual graph of the blue graph and vice versa.

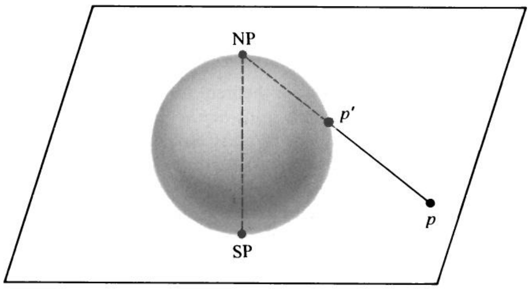


The red graph is the dual graph of the blue graph and vice versa.

Theorem 0.9.1 A graph has a dual if and only if it is planar.

Theorem 0.9.2 A graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

Proof. To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere. It is accomplished by stereographic projection of a sphere on a plane. Put the sphere on the plane and call the point of contact SP (south pole). At point SP, draw a straight line perpendicular to the plane, and let the point where this line intersects the surface of the sphere be called NP (north pole).



Stereographic projection.

Now, corresponding to any point p on the plane, there exists a unique point p' on the sphere and vice versa, where p' is the point at which the straight line from point p to point NP intersects the surface of the sphere. Thus there is a one-to-one correspondence between the points of the sphere and the finite points on the plane, and points at infinity in the plane correspond to the point NP on the sphere. From this construction, it is clear that any graph that can be embedded in a plane (i.e., drawn on a plane such that its edges do not intersect) can also be embedded in the surface of the sphere, and vice versa.

Hence, a graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane. ■

Unit 3

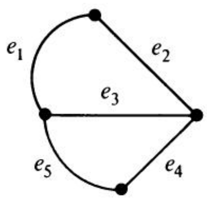
> **Theorem 0.9.3** The ring sum of two circuits in a graph G is either a circuit or an edge-disjoint union of circuits.

Proof. Let Γ_1 and Γ_2 be any two circuits in a graph G. If the two circuits have no edges or vertices in common, their ring sum $\Gamma_1 \bigoplus \Gamma_2$ is a disconnected subgraph of G, and is obviously an edge-disjoint union of circuits. If Γ_1 and Γ_2 do have edges and/or vertices in common, we have the following possible situations:
Since the degree of every vertex in a graph that is a circuit is two, every vertex v in subgraph $\Gamma_1 \bigoplus \Gamma_2$ has degree $d(v)$, where

$$\begin{cases} d(v) = 2, \text{ if } v \text{ is in } \Gamma_1 \text{ or } \Gamma_2 \text{ only; or if one of the edges formerly incident on } v \text{ was in both } \Gamma_1 \,\&\, \Gamma_2 \\ d(v) = 4, \text{ if } \Gamma_1 \text{ and } \Gamma_2 \text{ just intersect at } v \text{ (without a common edge).} \end{cases}$$

There is no other type of vertex in $\Gamma_1 \bigoplus \Gamma_2$. Thus $\Gamma_1 \bigoplus \Gamma_2$ is an Euler graph, and therefore consists of either a circuit or an edge-disjoint union of circuits. ■

0.10 VECTOR SPACE ASSOCIATED WITH A GRAPH



G

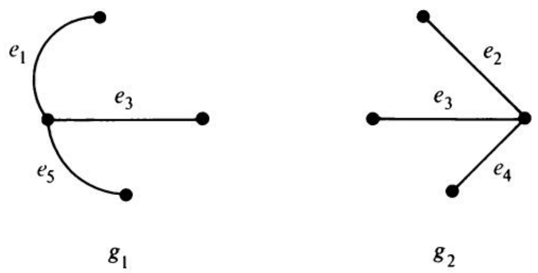
Consider the graph G with four vertices and five edges e_1, e_2, e_3, e_4, e_5. Any subset of these five edges (i.e., any subgraph g) of G can be represented by a 5-tuple:

$$X = (x_1, x_2, x_3, x_4, x_5)$$

such that

$$x_i = \begin{cases} 1 & \text{if } e_i \text{ is in } g \text{ and} \\ 0 & \text{if } e_i \text{ is not in } g \end{cases}$$

For example, the subgraph g_1 of G can be represented by $(1,0,1,0,1)$ and the subgraph g_2 of G can be represented by $(0,1,1,1,0)$.



Altogether there are $2^5 = 32$ such 5-tuples possible, including the zero vector $0 = (0,0,0,0,0)$, which represents a null graph and $(1,1,1,1,1)$, which is G itself.
It is easy to check the ring-sum operation between two subgraphs corresponds to the modulo 2 addition between the two 5-tuples representing the two subgraphs. For example, consider two subgraphs g_1 and g_2 of G. The ring sum $\Gamma_1 \bigoplus \Gamma_2 = \{e_1, e_2, e_4, e_5\}$ represented by $(1,1,0,1,1)$, which is clearly modulo 2 addition of the 5-tuples for g_1 and g_2.

0.11 MATRIX REPRESENTATION OF GRAPHS

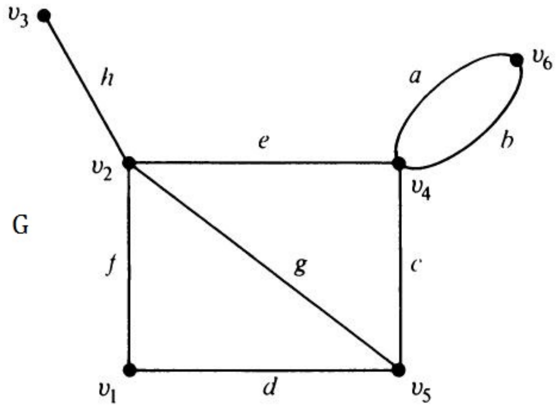
Definition 0.11.1 — INCIDENCE MATRIX. Let G be a graph with n vertices, e edges, and no self-loops. Define an n by e matrix $A = [a_{ij}]$, whose n rows correspond to the n vertices and the e columns correspond to the e edges, as follows:
The matrix element

$$a_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ edge } e_j \text{ is incident on } i^{th} \text{ vertex } v_i \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

R Incidence matrix is also called as the vertex-edge incidence matrix and is denoted by $A(G)$.

R The incidence matrix contains only two elements 0 and 1. Such a matrix is called a binary matrix or a $(0,1)$-matrix.

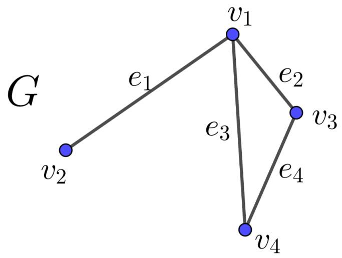
■ **Example 0.4** Write the incidence matrix of the following graph:



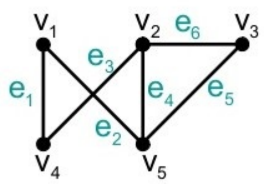
Solution:
Incidence matrix:

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

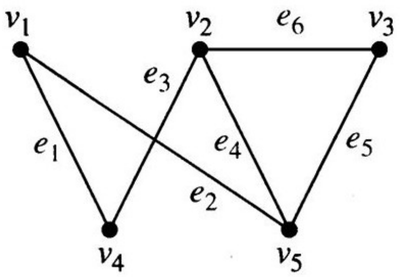
■



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$



	e1	e2	e3	e4	e5	e6
V1	1	1	0	0	0	0
V2	0	0	1	1	0	1
V3	0	0	0	0	1	1
V4	1	0	1	0	0	0
V5	0	1	0	1	1	0

R

1. Every edge is incident on exactly two vertices, each column of incident matrix has exactly two 1's.
2. The number of 1's in each row = the degree of the corresponding vertex.
3. A row with all O's, therefore, represents an isolated vertex.
4. Parallel edges in a graph produce identical columns in its incidence matrix.
5. If a graph G is disconnected and consists of two components g_1 and g_2, the incidence matrix $A(G)$ of graph G can be written in a block- diagonal form as

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$

where $A(g_1)$ and $A(g_2)$ are the incidence matrices of components g_1 and g_2.

Theorem 0.11.1 Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrices $A(G_1)$ and $A(G_2)$ differ only by permutations of rows and columns.

Theorem 0.11.2 If $A(G)$ is an incidence matrix of a connected graph G with n vertices, the rank of $A(G)$ is $n-1$.

Proof. Let $A(G)$ be an incidence matrix of a connected graph G with n vertices and m edges. Then

$$A(G) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}$$

Let the vectors in each row be respectively $A_1, A_2, A_3, \ldots.., A_n$. Then

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \cdots \\ \cdots \\ A_n \end{bmatrix}$$

Since there are exactly two 1's in every column of A, the sum (mod 2) of all these vectors are zero. Thus vectors $A_1, A_2, A_3, \ldots.., A_n$ are not linearly independent. Therefore, the rank of A is less than n,

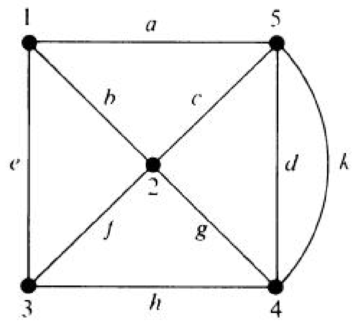
i.e., rank of $A(G) \leq n-1$.
Now, consider the sum of any k rows of these n vectors. $(k \leq n-1)$. If the graph is connected, $A(G)$ cannot be partitioned i.e., no $k \times k$ submatrix of A can be found for $k \leq n-1$. Hence, the addition of all vectors taken k at a time exhaust all possible linear combinations of $n-1$ row vectors, i.e., there is no linear combination of k row vectors of $A(G)$ equal to zero. So, the rank of $A(G)$ must be at least $n-1$. ■

Definition 0.11.2 Let $A(G)$ be an incidence matrix of a connected graph G with n vertices and e edges. If we delete any one row from $A(G)$, then the remaining $(n-1)$ by e submatrix A_f of A is called a reduced incidence matrix.

R The vertex corresponding to the deleted row is called the reference vertex.

R A tree is a connected graph with n vertices and $(n-1)$ edges, then its reduced incidence matrix is a square matrix of order $(n-1)$ by $(n-1)$.

■ **Example 0.5** Write the incidence matrix and reduced incidence matrix of the following graph:



Solution: ■

$$\mathsf{A} = \begin{array}{c} \begin{matrix} b & e & f & g & d & a & c & h & k \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Incidence matrix.

$$A_f = \begin{array}{c} \begin{matrix} b & e & f & g & d & a & c & h & k \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Reduced incidence matrix.

Corollary 0.11.3 The reduced incidence matrix of a tree is non-singular.

Proof. A graph with n vertices and $n-1$ edges that is not a tree is disconnected. The rank of the incidence matrix of such a graph will be less than $n-1$. Therefore, the $n-1$ by $n-1$ reduced incidence matrix of such a graph will not be nonsingular. In other words, the reduced incidence matrix of a graph is non-singular if and only if the graph is a tree. ■

R A matrix obtained by deleting (neglecting) some rows or columns of a matrix is said to be a **submatrix** of the given matrix.

Example:
If $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$, then a few submatrices of A are $[1]$, $[2]$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$, A.
But $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ are not submatrices of A.

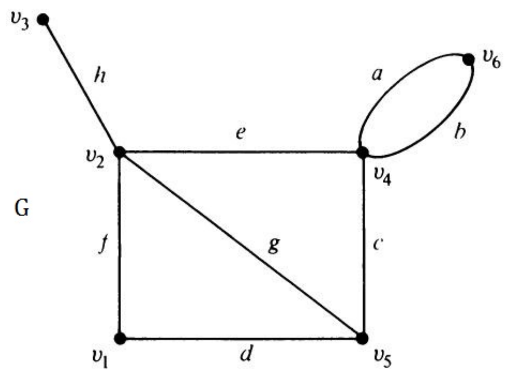
Theorem 0.11.4 Let $A(G)$ be an incidence matrix of a connected graph G with n vertices. An $n-1$ by $n-1$ submatrix of $A(G)$ is nonsingular if and only if the $n-1$ edges corresponding to the $n-1$ columns of this matrix constitute a spanning tree in G.

Definition 0.11.3 — CIRCUIT MATRIX. Let the number of different circuits in a graph G be q and the number of edges in G be e. Then a circuit matrix $B = [b_{ij}]$ of G is a q by e, $(0,1)$-matrix defined as follows:
The matrix element

$$b_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ circuit includes } j^{th} \text{ edge and} \\ 0, & \text{otherwise} \end{cases}$$

R Circuit matrix is denoted by $B(G)$.

■ **Example 0.6** Write the circuit matrix of the following graph:



Solution:
Circuit matrix: The graph G has four different circuits, $\{a,b\},\{c,e,g\},\{d,f,g\}$ and $\{c,d,f,e\}$. Therefore, its circuit matrix is a 4 by 8 matrix:

$$B(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

■

R

1. A column of all zeros corresponds to a noncircuit edge (i.e., an edge that does not belong to any circuit).
2. Each row of $B(G)$ is a circuit vector.
3. The number of 1's in a row is equal to the number of edges in the corresponding circuit.
4. If graph G is separable (or disconnected) and consists of two blocks (or components) g_1 and g_2, the circuit matrix $B(G)$ can be written in a block-diagonal form as

$$B(G) = \left[\begin{array}{c|c} B(g_1) & 0 \\ \hline 0 & B(g_2) \end{array}\right]$$

where $B(g_1)$ and $B(g_2)$ are the circuit matrices of components g_1 and g_2.

Theorem 0.11.5 Let B and A be respectively the circuit matrix and the incidence matrix (of a self-loop-free graph) whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row A;

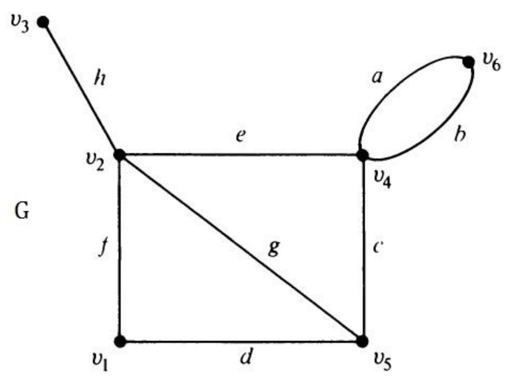
$$i.e.,\ \ AB^T = BA^T = 0\ (mod\ 2)$$

where superscript T denotes the transposed matrix.

Proof. Consider a vertex v and a circuit Γ in the graph G. Either v is in Γ or it is not. If v is not in Γ, there is no edge in the circuit Γ that is incident on v. On the other hand, if v is in Γ, the number of those edges in the circuit Γ that are incident on v is exactly two.
Now, consider the i^{th} row in A and the j^{th} row in B. Since the edges are arranged in the same order, the non-zero entries in the corresponding positions occur only if the particular edge is incident on the i^{th} vertex and is also in the j^{th} circuit.

If the i^{th} vertex is not in the j^{th} circuit, there is no such non-zero entry, and the dot product of the two rows is zero. If the i^{th} vertex is in the j^{th} circuit, there will be exactly two 1's in the sum of the products of individual entries. Since $1+1 = 0\,(mod\,2)$, the dot product of the two arbitrary rows-one from A and the other from B-is zero. Hence the theorem. ■

Example: For the following graph:



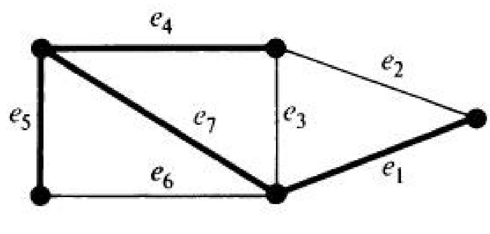
$$A\cdot B^T = \begin{bmatrix} 0&0&0&1&0&1&0&0\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&1\\ 1&1&1&0&1&0&0&0\\ 0&0&1&1&0&0&1&0\\ 1&1&0&0&0&0&0&0 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0\\ 1&0&0&0\\ 0&1&0&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&0&1&1\\ 0&1&1&0\\ 0&0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} \pmod 2.$$

Definition 0.11.4 — FUNDAMENTAL CIRCUIT MATRIX. A submatrix $B_f(G)$ of a circuit matrix $B(G)$ in which all rows correspond to a set of fundamental circuits w.r.to spanning tree in G is called a fundamental circuit matrix.

(R) Fundamental circuit matrix is denoted by B_f.

■ **Example 0.7** Write the fundamental circuit matrix with respect to the spanning tree (shown in heavy lines) of the following graph:



Solution:
Branches $= \{e_1, e_4, e_5, e_7\}$.
Chords $= \{e_2, e_3, e_6\}$. (There are 3 chords, therefore there must be 3 fundamental circuits).
Fundamental Circuits are:
① $= \{e_2, e_1, e_7, e_4\}$
② $= \{e_3, e_4, e_7\}$
③ $= \{e_6, e_5, e_7\}$
Fundamental circuit matrix:

	e_2	e_3	e_6	e_1	e_4	e_5	e_7
①	1	0	0	1	1	0	1
②	0	1	0	0	1	0	1
③	0	0	1	0	0	1	1

■

Theorem 0.11.6 (Sylvester's theorem) If Q is k by n matrix and R is an n by p matrix, then rank of $Q+$rank of $R<n$.

Theorem 0.11.7 If B is a circuit matrix of a connected graph G with e edges and n vertices, then rank of $B=e-n+1$.

Proof. If A is an incidence matrix of G, then we have $A\cdot B^T=O\,(mod\,2)$.
Therefore, according to Sylvester's theorem, rank of $A+$rank of $B<e$.

$$\implies \text{rank of } B<e-\text{rank of } A \tag{5}$$

Since rank of $A=n-1$, equation (5) becomes,

$$\text{rank of } B<e-(n-1) \implies \text{rank of } B<e-n+1 \tag{6}$$

But

$$\text{rank of } B>e-n+1 \tag{7}$$

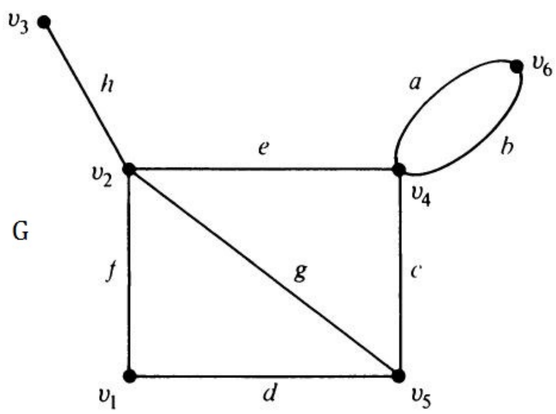
From equations (2) and (3), we get rank of $B=e-n+1$. ■

Definition 0.11.5 — CUTSET MATRIX. Let the number of rows represent the cutsets and the number of columns represent the edges of a graph G. Then the cutset matrix $C=[c_{ij}]$ of G is defined as follows:
The matrix element

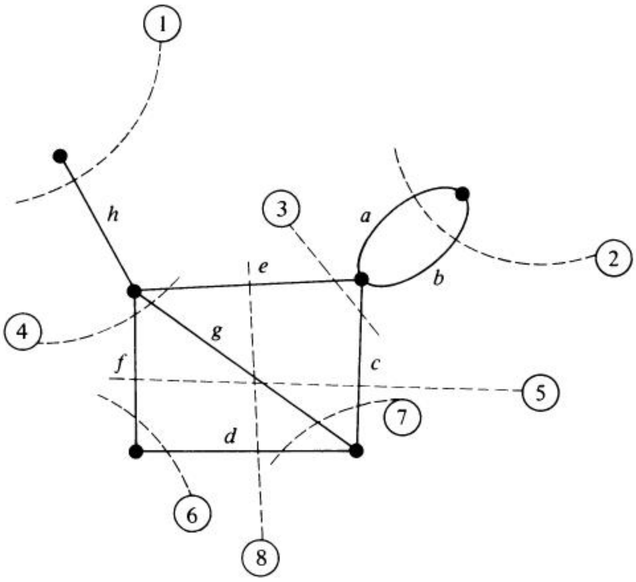
$$c_{ij}=\begin{cases}1, & \text{if } i^{th} \text{ cutset contains } j^{th} \text{ edge and}\\ 0, & \text{otherwise}\end{cases}$$

R Cutset matrix is denoted by $C(G)$.

■ **Example 0.8** Write the cutset matrix of the following graph:



Solution:



The graph G has 8 different cutsets, $\{h\}, \{a,b\}, \{e,c\}, \{f,g,e\}, \{f,g,c\}, \{d,f\}, \{c,g,d\}$, and {e,g,d}. Therefore, its cutset matrix is a 8 by 8 matrix:

C =

	a	b	c	d	e	f	g	h
1	0	0	0	0	0	0	0	1
2	1	1	0	0	0	0	0	0
3	0	0	1	0	1	0	0	0
4	0	0	0	0	1	1	1	0
5	0	0	1	0	0	1	1	0
6	0	0	0	1	0	1	0	0
7	0	0	1	1	0	0	1	0
8	0	0	0	1	1	0	1	0

■

R

1. Each row in $C(G)$ is a cut-set vector.
2. A column with all 0's corresponds to an edge forming a self-loop.
3. Parallel edges produce identical columns in the cut-set matrix.
4. The rank of cut-set matrix $C(G)$ = the rank of the incidence matrix $A(G)$ = the rank of graph G.
5. Since the number of edges common to a cut-set and a circuit is always even, every row in C is orthogonal to every row in B, provided the edges in both B and C are arranged in the same order. i.e.,

$$B \cdot C^T = C \cdot B^T = 0 \,(mod\ 2)$$

Theorem 0.11.8 The rank of cut-set matrix $C(G)$ is equal to the rank of the incidence matrix $A(G)$, which equals the rank of graph G.

Proof. Since the matrix $A(G)$ is a submatrix of cut-set matrix $C(G)$, therefore

$$\text{rank of } C(G) \geq \text{rank of } A(G) = n-1. \tag{1}$$

We know that $B(G)\, C(G)^T = O$.
Since G is a connected graph, by Sylvestor's theorem,

$$\text{rank of } B(G) + \text{rank of } C(G) \leq e.$$

Also,

$$\begin{aligned}\text{rank of } B(G) &= e-n+1\\ \text{rank of } C(G) &\leq e - \text{rank of } B(G)\\ &\leq e-(e-n+1)\end{aligned}$$

$$\text{rank of } C(G) \leq n-1. \tag{2}$$

Combining (1) and (2), we get

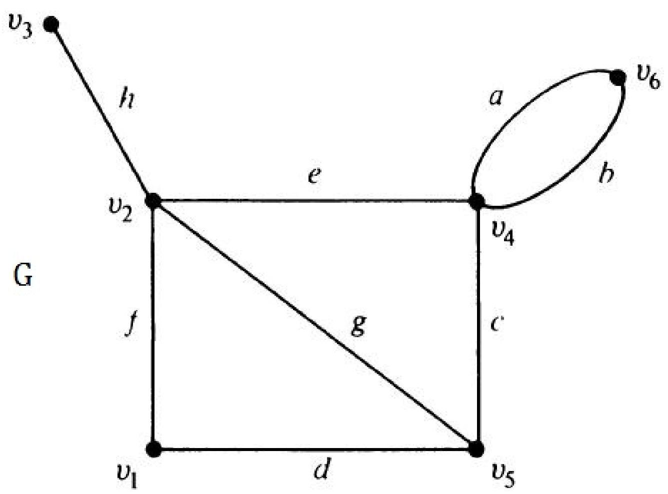
$$\text{rank of } C(G) = n-1 = \text{rank of } G.$$

■

Definition 0.11.6 — PATH MATRIX. A path matrix is defined for a specific pair of vertices in a graph, say (x,y), and is written as $P(x,y)$. The rows in $P(x,y)$ correspond to different paths between vertices x and y, and the columns correspond to the edges in G. That is, the path matrix for (x,y) vertices is $P(x,y) = [p_{ij}]$, where

$$p_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ edge lies in } i^{th} \text{ path and} \\ 0, & \text{otherwise} \end{cases}$$

■ **Example 0.9** Write the path matrix between vertices v_3 and v_4 of the following graph:



Solution: There are three different paths:

1. $\{h, e\}$
2. $\{h, g, c\}$
3. $\{h, f, d, c\}$

Therefore, its path matrix is a 3 by 8 matrix:

$$P(v_3, v_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

■

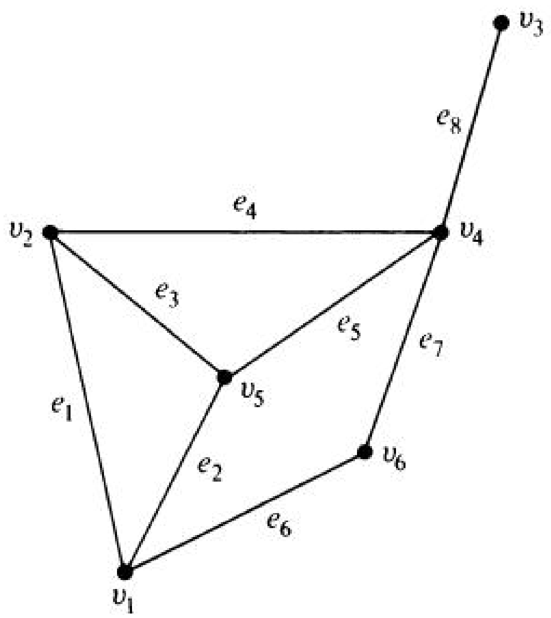
R

1. A column of all 0's corresponds to an edge that does not lie in any path between x and y.
2. A column of all 1's corresponds to an edge that lies in every path between x and y.
3. There is no row with all 0's.
4. The ring sum of any two rows in $P(x, y)$ corresponds to a circuit or an edge-disjoint union of circuits.

Definition 0.11.7 — ADJACENCY MATRIX. The adjacency (connection) matrix of a graph G with n vertices and no parallel edges is an n by n symmetric binary matrix $X = [x_{ij}]$ defined over the ring of integers such that

$$x_{ij} = \begin{cases} 1, & \text{if there is an edge between } i^{th} and\, j^{th} vertices, and \\ 0, & \text{if there is no edge between them.} \end{cases}$$

■ **Example 0.10** Write the adjacency matrix of the following graph:



Solution:
Adjacency matrix:

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

■

R

1. The entries along the principal diagonal of X are all O's if and only if the graph has no self-loops. A self-loop at the i^{th} vertex corresponds to x_{ii}.
2. If the graph has no self-loops (and no parallel edges), the degree of a vertex equals the number of l's in the corresponding row or column of X.
3. If graph G is separable (or disconnected) and consists of two blocks (or components) g_1 and g_2, the adjacency matrix $X(G)$ can be written in a block-diagonal form as

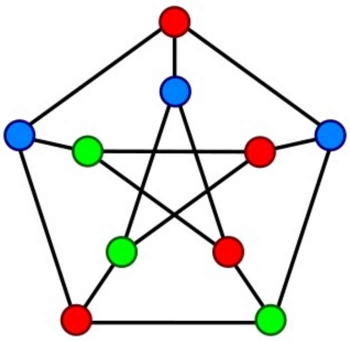
$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right]$$

where $X(g_1)$ and $X(g_2)$ are the adjacency matrices of components g_1 and g_2.

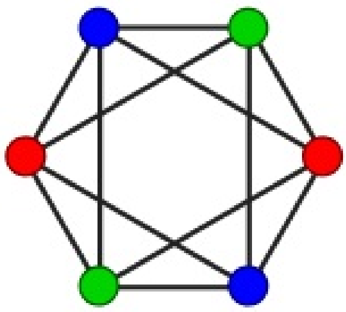
Unit 4

> **Definition 0.11.8 — PROPER COLOURING.** Painting all the vertices of a graph with colours in such a way that no two adjacent vertices have the same colour is called the proper colouring of a graph.

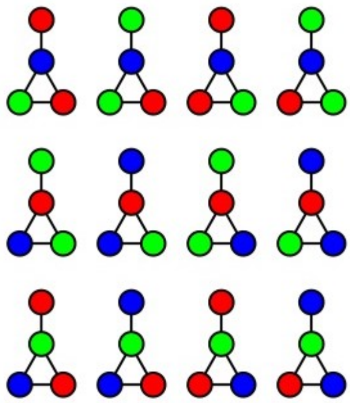
Examples:



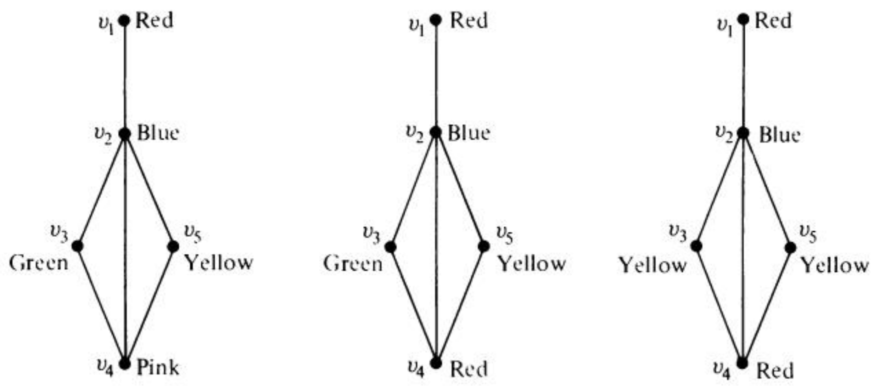
A proper vertex colouring of the **Petersen graph** with 3 colours.



A proper vertex colouring of the graph with 3 colours.



This graph can be 3-coloured in 12 different ways.



Proper colourings of a graph in 3 different ways.

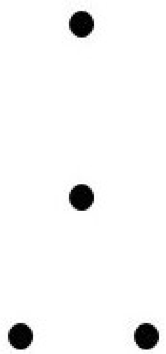
R A graph in which every vertex has been assigned a colour according to a proper colouring is called a properly coloured graph.

R A given graph can be properly coloured in many different ways.

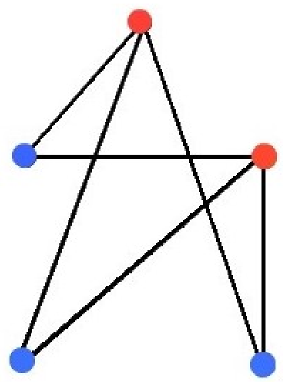
R For colouring problems we have to consider only simple, connected graphs.

Definition 0.11.9 — CHROMATIC NUMBER. A graph G that requires κ different colours for its proper colouring, and no less, is called a κ-chromatic graph, and the number κ is called the chromatic number of G.

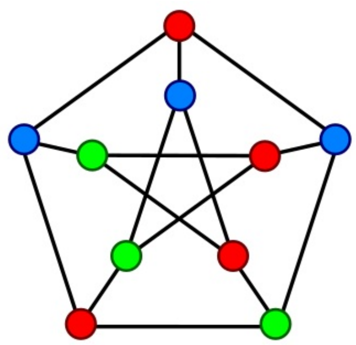
Examples:



1-chromatic graph



2-chromatic graph



3-chromatic graph

R All parallel edges between two vertices can be replaced by a single edge without affecting adjacency of vertices.

R Self-loops must be ignored.

R A graph consisting of only isolated vertices is 1-chromatic.

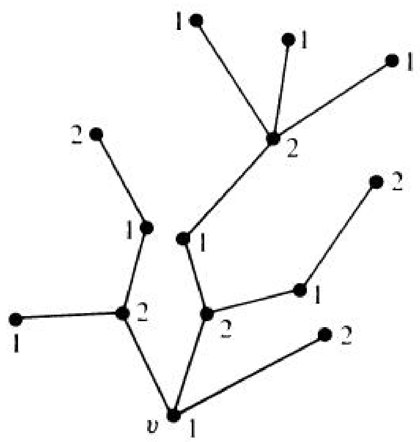
R A graph with one or more edges is at least 2-chromatic (or bi-chromatic).

R A complete graph with n vertices is n-chromatic. Hence a graph containing a complete graph of r vertices is at least r-chromatic.

R A graph consisting of simply one circuit with $n > 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd.

Theorem 0.11.9 Every tree with two or more vertices is 2-chromatic.

Proof. Select any vertex v in the given tree T. Consider T as a rooted tree at vertex v. Paint v with colour 1. Paint all vertices adjacent to v with colour 2. Next, paint the vertices adjacent to these using colour 1. Continue this process till every vertex in T has been painted. Now in T we find that all vertices at odd distances from v have colour 2, while v and vertices at even distances from v have colour 1.



Proper colouring of a tree.

Now along any path in T the vertices are of alternating colours. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same colour. Thus T has been properly coloured with two colours. One colour is not sufficient. Hence, T is 2-chromatic. ■

Theorem 0.11.10 A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

Proof. Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G. Using the colouring procedure and we know that "Every tree with two or more vertices is 2-chromatic", let us properly colour T with two colours. Now add the chords to T one by one. Since G had no circuits of odd length, the end vertices of every chord being replaced are differently coloured in T. Thus G is coloured with two colours, with no adjacent vertices having the same colour. That is, G is 2-chromatic. Conversely, if G has a circuit of odd length, we would need at least three colours just for that circuit . Hence the theorem. ■

Theorem 0.11.11 If $d_{\max}$ is the maximum degree of the vertices in a graph G, then chromatic number of $G \leq 1 + d_{\max}$.

Definition 0.11.10 — CHROMATIC POLYNOMIAL. A given graph G of n vertices can be properly coloured in many different ways using a sufficiently large number of colours. This property of a graph is expressed in a simple manner of a polynomial. This polynomial is called the chromatic polynomial of G and is defined as follows:
The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly colouring the graph, using λ or lesser colours.
Let c_i be the different ways of properly colouring G using exactly i different colours. Since i colours can be selected out of λ colours in

$$\binom{\lambda}{i} \text{ different ways,}$$

there are $c_i \binom{\lambda}{i}$ different ways of properly colouring G using exactly i colours out of λ colours.
Since i can be any positive integer from 1 to n (it is not possible to use more than n colours upon n vertices), the chromatic polynomial is a sum of these terms; i.e.,

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \binom{\lambda}{i}$$

i.e.,

$$P_n(\lambda) = c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots + c_n \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}$$

Each c_i has to be evaluated individually for the given graph.

R Any graph with even one edge requires at least two colours for proper colouring $\implies c_1 = 0$.

R A graph with n vertices and using n different colours can be properly coloured in $n!$ ways $\implies c_n = 0$.

Theorem 0.11.12 A graph of n vertices is a complete graph if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)\cdots(\lambda-n+1)$$

Proof. With λ colors, there are λ different ways of colouring any selected vertex of a graph. A second vertex can be colored properly in exactly $\lambda-1$ ways, the third vertex in $\lambda-2$ ways, the fourth vertex in $\lambda-3$ ways, . . . , and the n^{th} vertex in $\lambda-n+1$ ways if and only if every vertex is adjacent to every other. That is, if and only if the graph is complete. ■

Theorem 0.11.13 An n-vertex graph is a tree if and only if its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda-1)^{n-1}$$

Proof. We can prove this theorem by induction on n.
When $n = 1$, there is only one vertex, which can be coloured in λ ways.
$\therefore P_1(\lambda) = \lambda(\lambda-1)^0$, which is true.
Assuming that the result is true for all trees with $n \leq m$.
Consider a tree T with m vertices.
We know that every tree has at least two pendant vertices.
Remove one pendant vertex from T to get a tree T' with $m-1$ vertices.
Applying induction assumption, T' can be coloured with $\lambda(\lambda-1)^{m-1}$.
Now attach the removed pendant vertex to T'. Since this vertex is adjacent to only one vertex of T', it can be coloured with $\lambda-1$ ways.
Thus T can be coloured with $\lambda(\lambda-1)^{m-2}(\lambda-1) = \lambda(\lambda-1)^{m-1}$ ways.
Hence for any n, $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$. ■

Theorem 0.11.14 Let a and b be two non-adjacent vertices in a graph G. Let G' be a graph obtained by adding an edge between a and b. Let G'' be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with single edge. Then

$$P_n(\lambda) = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''$$

.

Proof. The number of ways of properly coloring G can be grouped into two cases, one such that vertices a and b are of the same color and the other such that a and b are of different colors.
Since the number of ways of properly coloring G such that a and b have different colors = number of ways of properly coloring G', and number of ways of properly coloring G such that a and b have the same color = number of ways of properly coloring G'',

$$P_n(\lambda) = P_n(\lambda) \text{ of } G' + P_{n-1}(\lambda) \text{ of } G''$$

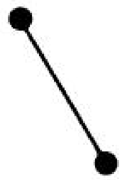
. ■

R This theorem is one of the best techniques in evaluating the chromatic polynomial of a given graph.

R To evaluate chromatic polynomial of the given graph G with n vertices:

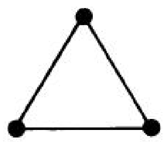
If G is	then
Complete graph	$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)\cdots(\lambda-n+1)$
Tree	$P_n(\lambda) = \lambda(\lambda-1)^{n-1\}}$
neither complete graph nor tree	Use Theorem 0.11.14

■ **Example 0.11** Evaluate chromatic polynomial of the following graph:



Solution: Since given graph is a complete graph with 2 vertices (K_2).
∴ Its chromatical polynomial is $P_2(\lambda) = \lambda(\lambda-1)$. ■

■ **Example 0.12** Evaluate chromatic polynomial of the following graph:

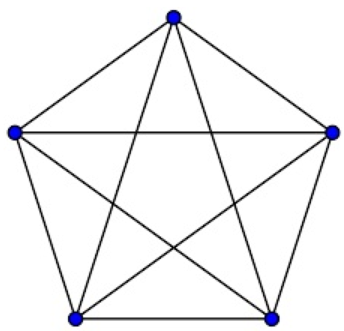


Solution: Since given graph is a complete graph with 3 vertices (K_3).
∴ Its chromatical polynomial is $P_3(\lambda) = \lambda(\lambda-1)(\lambda-2)$. ■

■ **Example 0.13** Evaluate chromatic polynomial of the following graph:

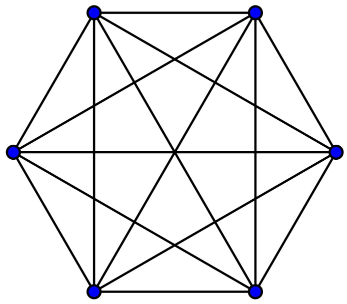
Solution: Since given graph is a complete graph with 4 vertices (K_4).
∴ Its chromatical polynomial is $P_4(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$. ■

■ **Example 0.14** Evaluate chromatic polynomial of the following graph:



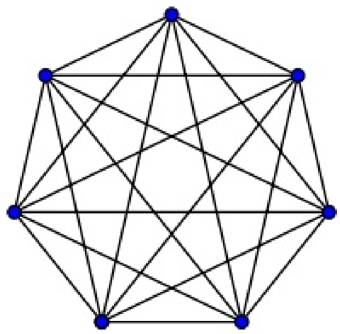
Solution: Since given graph is a complete graph with 5 vertices (K_5).
∴ Its chromatical polynomial is $P_5(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$. ■

■ **Example 0.15** Evaluate chromatic polynomial of the following graph:



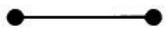
Solution: Since given graph is a complete graph with 6 vertices (K_6).
$\therefore$ Its chromatical polynomial is $P_6(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)$. ■

■ **Example 0.16** Evaluate chromatic polynomial of the following graph:



Solution: Since given graph is a complete graph with 7 vertices (K_7).
$\therefore$ Its chromatical polynomial is $P_7(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)$. ■

■ **Example 0.17** Evaluate chromatic polynomial of the following graph

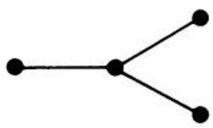


Solution: Since given graph is a tree with 2 vertices.
$\therefore$ Its chromatical polynomial is $P_2(\lambda) = \lambda(\lambda-1)$. ■

■ **Example 0.18** Evaluate chromatic polynomial of the following graph

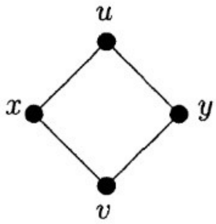
Solution: Since given graph is a tree with 3 vertices.
$\therefore$ Its chromatical polynomial is $P_3(\lambda) = \lambda(\lambda-1)^2$. ■

■ **Example 0.19** Evaluate chromatic polynomial of the following graph:

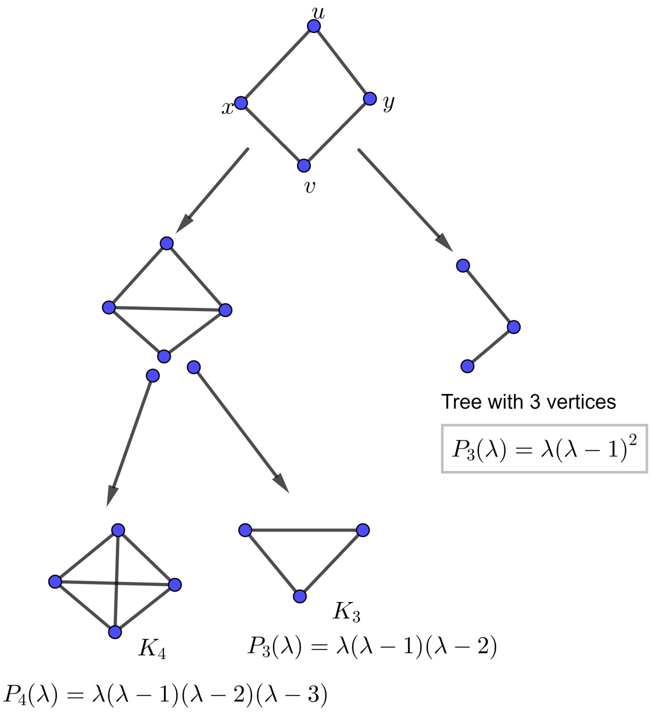


Solution: Since given graph is a tree with 4 vertices.
$\therefore$ Its chromatical polynomial is $P_4(\lambda) = \lambda(\lambda-1)^3$. ■

■ **Example 0.20** Evaluate chromatic polynomial of the following graph:



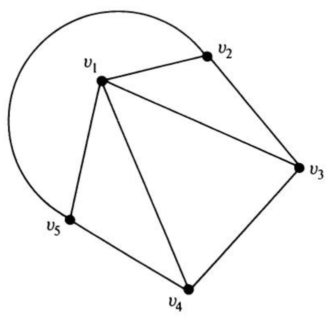
Solution:



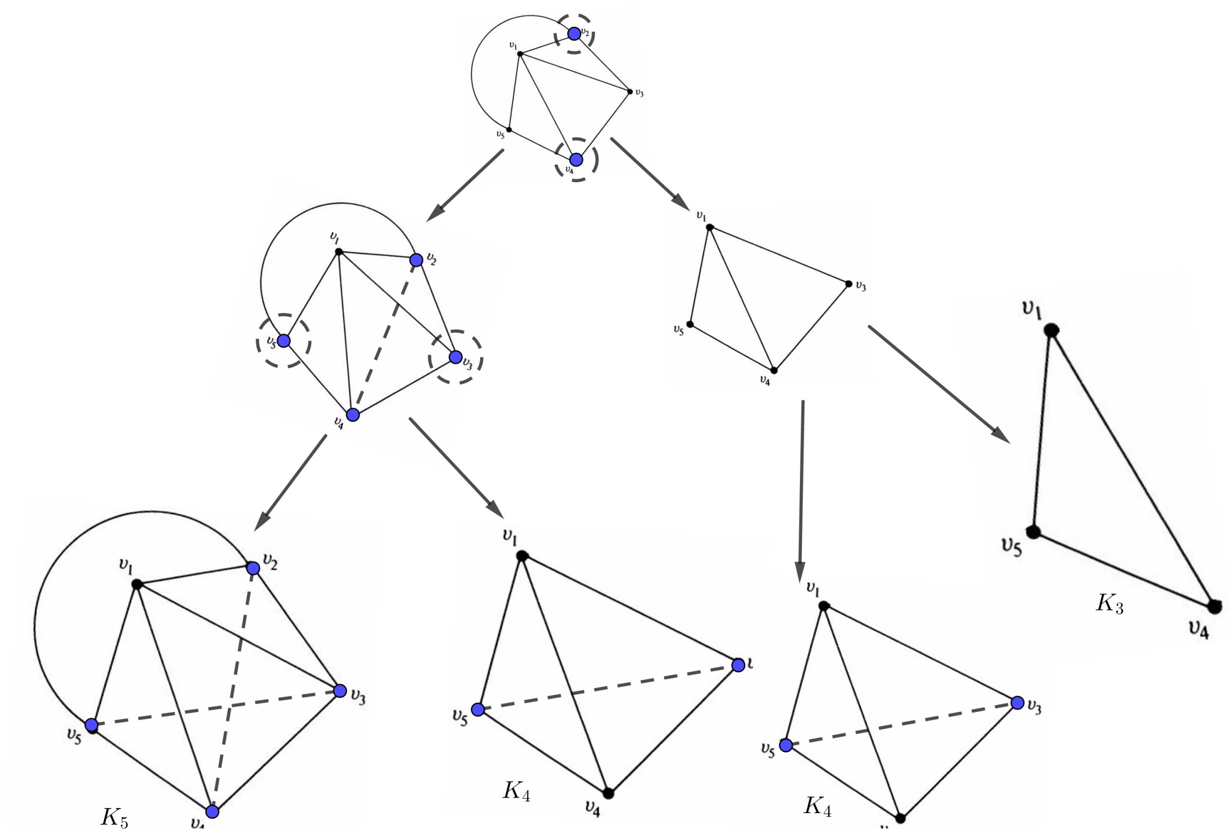
$$
\begin{aligned}
\therefore P_n(\lambda) &= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)^2 \\
&= \lambda(\lambda-1)[(\lambda-2)(\lambda-3) + (\lambda-2) + (\lambda-1)] \\
&= \lambda(\lambda-1)[\lambda^2 - 5\lambda + 6 + \lambda - 2 + \lambda - 1] \\
&= \lambda(\lambda-1)(\lambda^2 - 3\lambda + 3)
\end{aligned}
$$

■

■ **Example 0.21** Evaluate chromatic polynomial of the following graph:



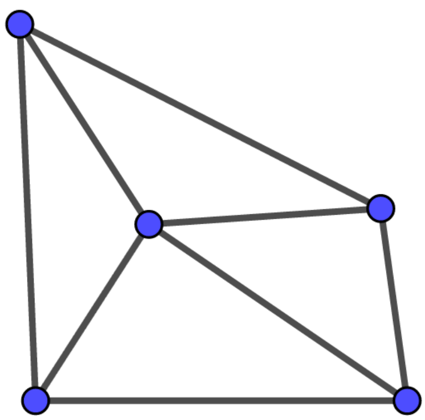
Solution:



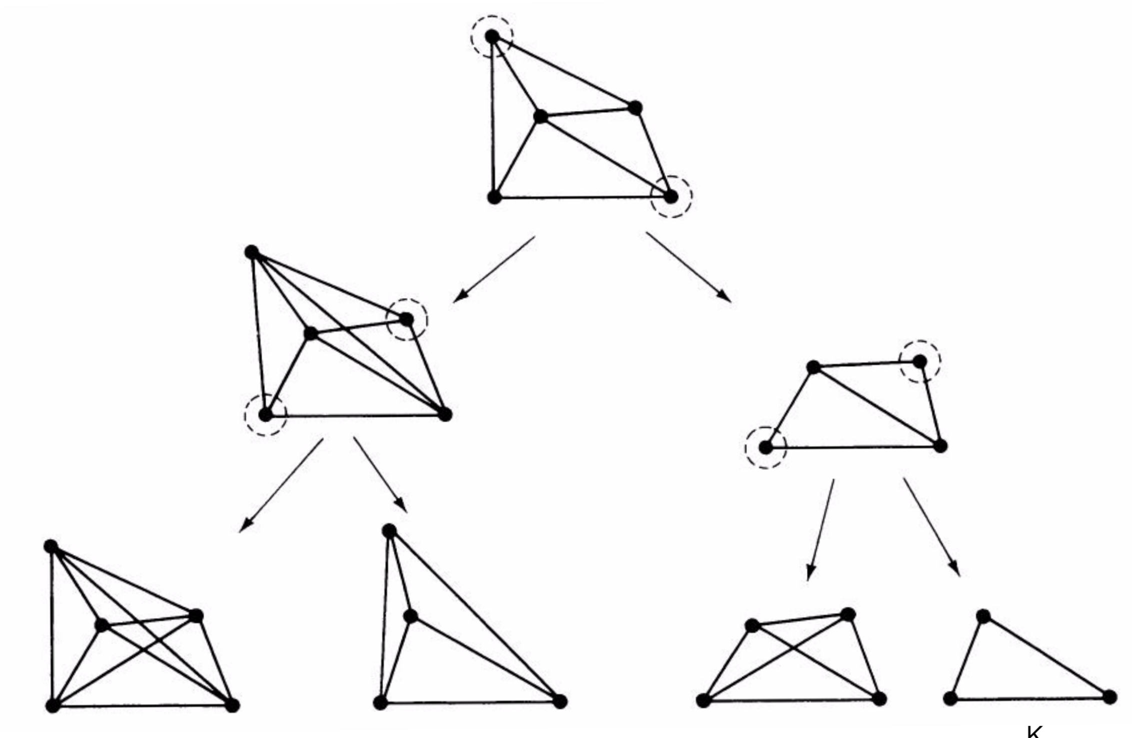
$$\begin{aligned}
\therefore P_n(\lambda) &= P_n(\lambda) \text{ of } K_5 + 2\cdot P_n(\lambda) \text{ of } K_4 + P_n(\lambda) \text{ of } K_3 \\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) \\
&= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1] \\
&= \lambda(\lambda-1)(\lambda-2)[\lambda^2 - 7\lambda + 12 + 2\lambda - 6 + 1] \\
&= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)
\end{aligned}$$

■

■ **Example 0.22** Evaluate chromatic polynomial of the following graph:



Solution:



$$
\begin{aligned}
\therefore P_n(\lambda) &= P_n(\lambda) \text{ of } K_5 + 2 \cdot P_n(\lambda) \text{ of } K_4 + P_n(\lambda) \text{ of } K_3 \\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) \\
&= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1] \\
&= \lambda(\lambda-1)(\lambda-2)[\lambda^2 - 7\lambda + 12 + 2\lambda - 6 + 1] \\
&= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)
\end{aligned}
$$

■

Unit 5

> **Definition 0.11.11 — DIRECTED GRAPH.** A directed graph (or a digraph) G consists of a set of vertices $V = \{v_1, v_2, v_3, \ldots\}$, a set of edges $E = \{e_1, e_2, e_3, \ldots\}$ and a mapping ψ that maps every edge onto some ordered pair of vertices (v_i, v_j).

R A digraph is also called as an oriented graph.

R In a digraph, an edge is not only incident into a vertex but is also incident out of a vertex. The vertex v_i, in which an edge e_k is incident out of, is called the initial vertex of e_k. The vertex v_j, in which an edge e_k is incident into, is called the terminal vertex of e_k.

R An edge for which the initial and terminal vertices are same is called a self-loop.

R The number of edges incident out of a vertex v_i is called the out-degree (or out-valence) of v_i and is written $d^+(v_i)$. The number of edges incident into v_i is called the in-degree (or in-valence) of v_i and is written as $d^-(v_i)$.

R In any digraph G, the sum of all out-degrees is equal to the sum of all in-degrees and each sum being equal to the number of edges in G; i.e.,

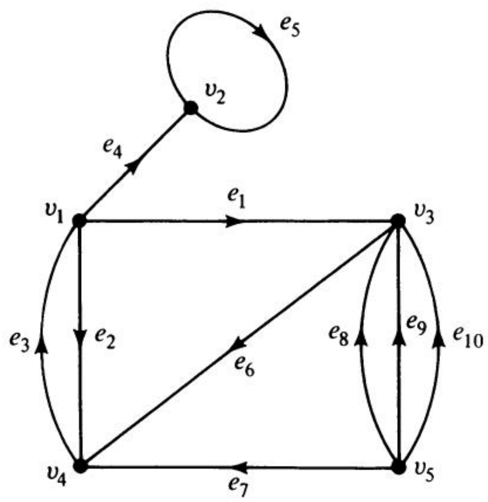
$$\boxed{\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i)}.$$

R An isolated vertex is a vertex in which the out-degree and the in-degree are both equal to zero. i.e., $d^+(v_i) = d^-(v_i) = 0$.

R A vertex in a digraph is called pendant if it is of degree one. i.e., $d^+(v_i) + d^-(v_i) = 1$.

R Two or more directed edges are said to be parallel if they are mapped onto the same ordered pair of vertices and must have same sense of direction.

Example:



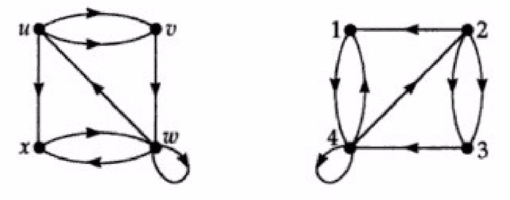
Directed graph with 5 vertices and 10 edges.

In this graph, v_3 is the initial vertex and v_4 is the terminal vertex of the edge e_6. An edge e_5 is the self-loop. Edges e_8, e_9 and e_{10} are parallel but edges e_2 and e_3 are not parallel.

$d^+(v_1) = 3;$ $\quad$ $d^-(v_1) = 1;$
$d^+(v_2) = 1;$ $\quad$ $d^-(v_2) = 2;$
$d^+(v_3) = 1;$ $\quad$ $d^-(v_3) = 4;$
$d^+(v_4) = 1;$ $\quad$ $d^-(v_4) = 3;$
$d^+(v_5) = 4;$ $\quad$ $d^-(v_5) = 0.$

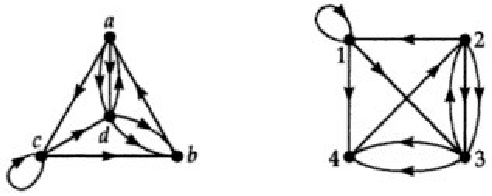
Definition 0.11.12 — ISOMORPHIC DIGRAPHS. Two digraphs G_1 and G_2 are said to be isomorphic if G_2 can be obtained by relabelling the vertices of G_1 i.e., if there is a one-one correspondence between the vertices of G_1 and those of G_2, such that the edges joining each pair of vertices in G_1 agree in both number and direction with the edges joining the corresponding pair of vertices in G_2.

Examples:

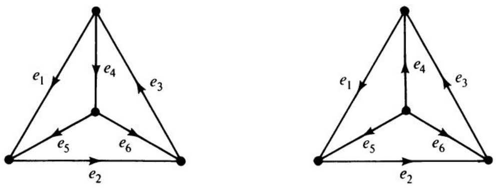


G_1 G_2

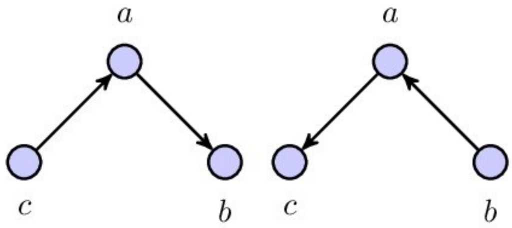
G_1 are G_2 isomorphic digraphs.



Two isomorphic digraphs.

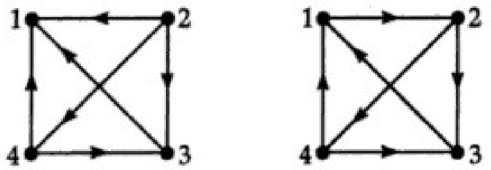


Two non-isomorphic digraphs.



Two non-isomorphic digraphs.

Two non-isomorphic digraphs.

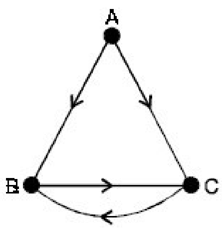


Two non-isomorphic digraphs.

0.11.1 Different Types of Digraphs:

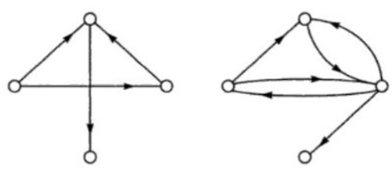
Definition 0.11.13 — SIMPLE DIGRAPH. A digraph that has neither self-loop nor parallel edges is called a simple digraph.

Example:



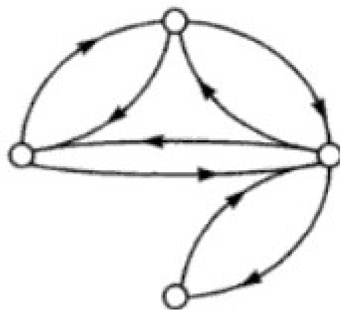
Definition 0.11.14 — ASYMMETRIC DIGRAPH. Digraphs that have at most one directed edge between a pair of vertices, but are allowed to have self-loops, are called asymmetric or antisymmetric.

Examples:



Definition 0.11.15 — SYMMETRIC DIGRAPH. Digraph in which for every edge (a,b) (i.e., from vertex a to b) there is also an edge (b,a) (i.e., from vertex b to a) is called a symmetric digraph.

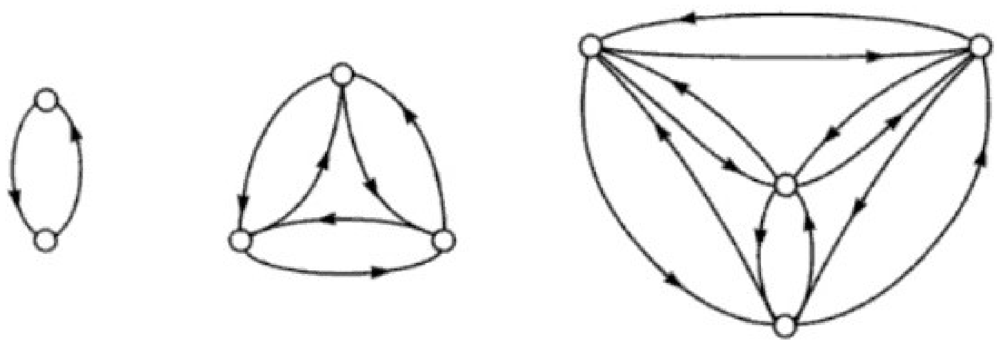
Example:



Definition 0.11.16 — COMPLETE SYMMETRIC DIGRAPH. A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex.

R A complete symmetric digraph of n vertices contains $n(n-1)$ edges.

Examples:



Definition 0.11.17 — COMPLETE ASYMMETRIC DIGRAPH. A complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices.

R A complete asymmetric digraph of n vertices contains $\frac{n(n-1)}{2}$ edges.

Definition 0.11.18 — BALANCED DIGRAPH. A digraph is said to be balanced if for every vertex v_i, the out-degree equals the in-degree; i.e., $d^+(v_i) = d^-(v_i)$.

Definition 0.11.19 — REGULAR DIGRAPH. A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.

Definition 0.11.20 — DIRECTED WALK. A directed walk from a vertex v_i to v_i is an alternating sequence of vertices and edges, beginning with v_i and ending with v_j, such that each edge is oriented from the vertex preceding it to the vertex following it.

R In a directed walk, no edge can appear more than once but vertices can appear any number of times.

R A directed walk which begins and ends in the same vertex is called a closed walk.

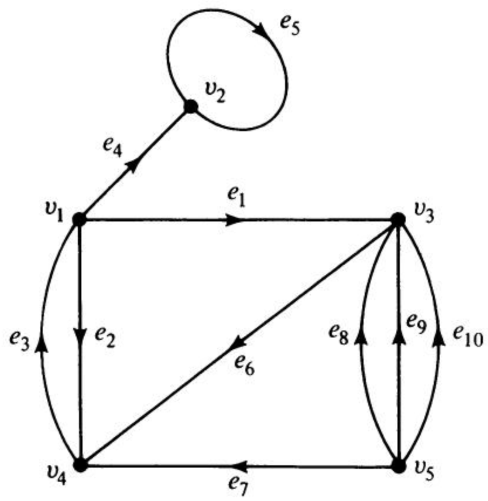
R A directed walk, in which the end vertices are different, is called an open walk.

R A directed path is a open directed walk in which no vertex appears more than once.

R A closed directed walk in which no vertex appears more than once (except initial and terminal vertices) is called a directed circuit.

R A semi-directed circuit in a directed graph is a circuit in the corresponding undirected graph but is not a directed circuit.

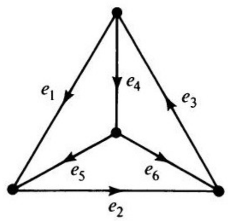
Example:



The sequence of vertices and edges $v_5e_8v_3e_6v_4e_3v_1$ is a directed path from v_5 to v_1, whereas $v_5e_7v_4e_6v_3e_1v_1$ has no such consistent direction from v_5 to v_1 and is a semi-path.
The sequence of vertices and edges $v_1e_1v_3e_6v_4e_3$ is a directed circuit, whereas $v_1e_1v_3e_6v_4e_3$ is a semi-directed circuit.

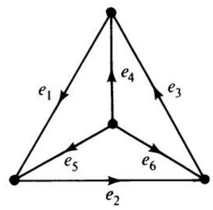
Definition 0.11.21 — STRONGLY CONNECTED DIGRAPH. A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.

Example:



Definition 0.11.22 — WEAKLY CONNECTED DIGRAPH. A digraph G is said to be weakly connected if its corresponding undirected graph is connected but G is not strongly connected.

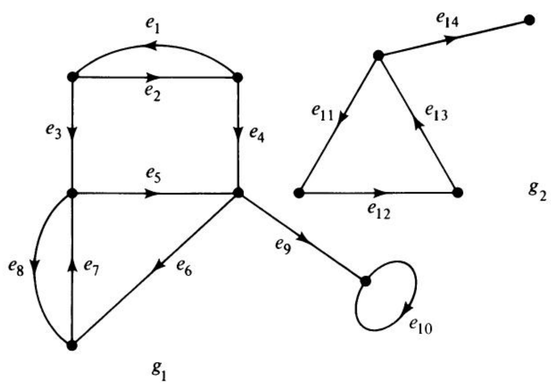
Example:



R A digraph G is connected means G may be strongly or weakly connected. A digraph that is not connected is called as disconnected.

R Each maximal connected (weakly or strongly) subgraph of a digraph G is said to be a component of G. But within each component of G the maximal strongly connected subgraphs are said to be fragments of G.

Example:



Disconnected digraph with two components.

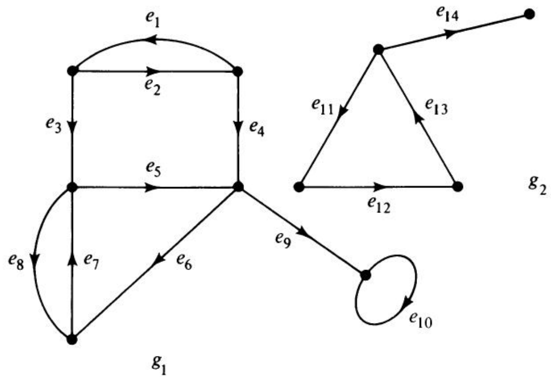
The component g_1 contains three fragments $\{e_1, e_2\}$, $\{e_5, e_6, e_7, e_8\}$, and $\{e_{10}\}$. Here edges e_3, e_4 and e_9 do not appear in any fragments of g_1.

Definition 0.11.23 — CONDENSATION OF A DIGRAPH. The condensation G_c of a digraph G is a digraph in which each strongly connected fragment is replaced by a vertex and all directed edges from one strongly connected component to another are replaced by a single directed edge.

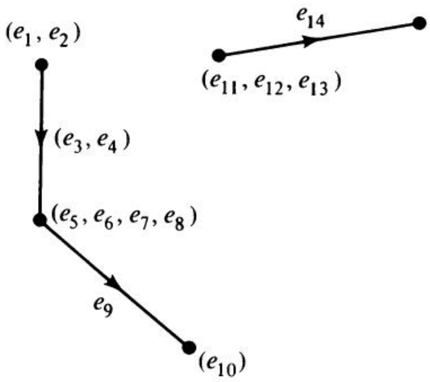
R The condensation of a strongly connected digraph is just a vertex.

R The condensation of a digraph has no directed circuit.

Example 0.23 Find the condensation of the following graph:



Solution:



■

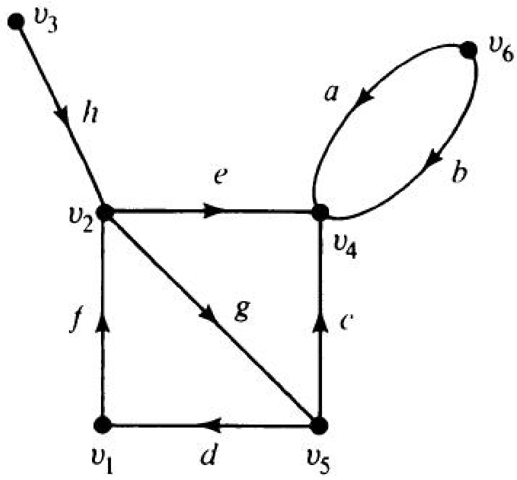
Definition 0.11.24 — ACCESSIBILITY. In a digraph a vertex b is said to be accessible (or reachable) from vertex a if there is a directed path from a to b.

R A digraph G is strongly connected if and only if every vertex in G is accessible from every other vertex.

Definition 0.11.25 — INCIDENCE MATRIX OF A DIGRAPH. The incidence matrix of a digraph with n vertices, e edges and no self-loops is an n by n matrix $A = [a_{ij}]$, whose rows correspond to vertices and columns correspond to edges, such that

$$a_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ edge is incident out of } i^{th} \text{ vertex } v_i \text{ and} \\ -1, & \text{if } j^{th} \text{ edge is incident into } i^{th} \text{ vertex } v_i \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

■ **Example 0.24** Write the incidence matrix for the following digraph:



Solution:

	a	b	c	d	e	f	g	h
v_1	0	0	0	−1	0	1	0	0
v_2	0	0	0	0	1	−1	1	−1
v_3	0	0	0	0	0	0	0	1
v_4	−1	−1	−1	0	−1	0	0	0
v_5	0	0	1	1	0	0	−1	0
v_6	1	1	0	0	0	0	0	0

■

Theorem 0.11.15 If $A(G)$ is the incidence matrix of a connected digraph of n vertices, then the rank of $A(G) = n - 1$.

Theorem 0.11.16 The determinant of every square submatrix of the incidence matrix of a digraph A is $1, -1$ or 0.

Proof. The theorem can be proved by expanding the determinant of a square submatrix of A. Consider a k by k submatrix M of A. If M has any column or row consisting of all zeros, det M is clearly zero. Also det $M = 0$ if every column of M contains two non-zero entries, 1 and -1. Now if det $M \neq 0$ (i.e., M is non-singular matrix), then the sum of entries in each column of M cannot be zero. Therefore, M must have a column in which there is a single non-zero element that is either 1 or -1. Let this single element be in the $(i, j)^{\text{th}}$ position in M. Thus

$$\det M = \pm 1 \cdot M_{ij}$$

where M_{ij} is the submatrix of M with its i^{th} row and j^{th} column deleted. The $(k-1)$ by $(k-1)$ submatrix M_{ij} is also nonsingular (because M is nonsingular); therefore, it must also have at least one column with a single non-zero entry, say in the $(p,q)^{\text{th}}$ position. Expanding det M_{ij} about this element in the $(p,q)^{\text{th}}$ position,

$$\det M_{ij} = \pm[\text{determinant of a non-singular } (k-2) \text{ by } (k-2) \text{ submatrix of } M].$$

Repeated application of this procedure, we get

$$\det M_{ij} = \pm 1.$$

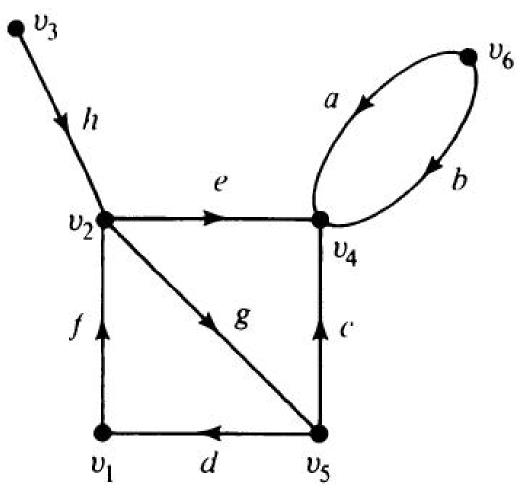
Hence the theorem. ■

R Any matrix with every square submatrix having a determinant of $1, -1$ or 0 is called a **unimodular matrix**.

Definition 0.11.26 — CIRCUIT MATRIX OF A DIGRAPH. Let G be a digraph with e edges and q circuits (directed circuits or semicircuits). An arbitrary orientation (clockwise or counterclockwise) is assigned to each of the q circuits. Then a circuit matrix $= [b_{ij}]$ of the digraph G is a q by e matrix defined as

$$b_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{circuit includes } j^{th} \text{ edge and the orientations of the edge} \\ & \text{and circuit coincide,} \\ -1, & \text{if } i^{th} \text{circuit includes } j^{th} \text{ edge but the orientations of the edge} \\ & \text{and circuit are opposite,} \\ 0, & \text{if } i^{th} \text{circuit does not include } j^{th} \text{edge.} \end{cases}$$

■ **Example 0.25** Write the circuit matrix for the following digraph:



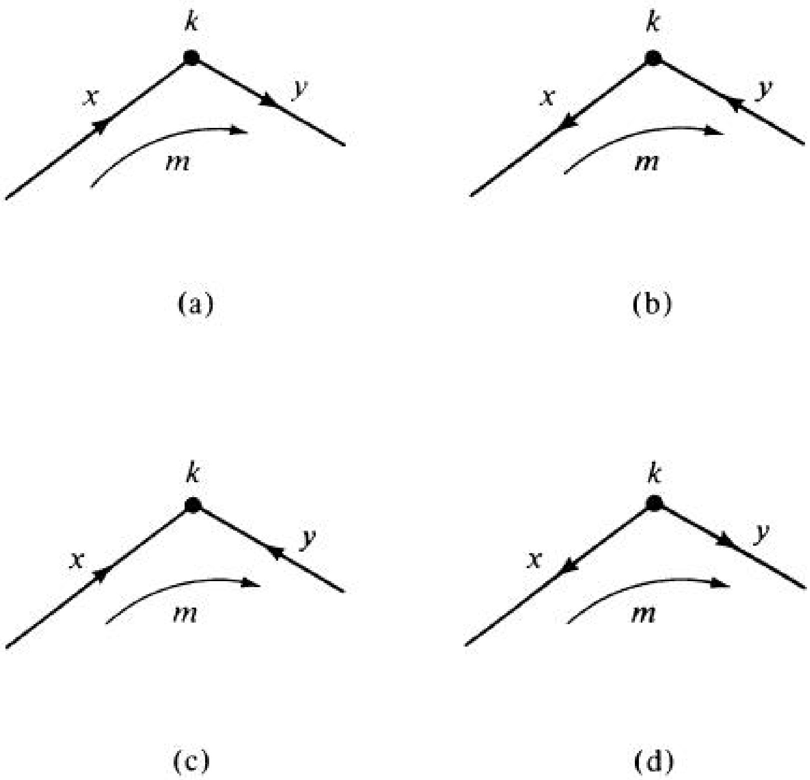
Solution: The graph has four different directed circuits:
①.$\{d, f, g\}$
②.$\{c, e, g\}$
③.$\{c, d, f, e\}$
④.$\{a, b\}$.
Therefore, its directed circuit matrix is a 4 by 8 matrix:

$$\begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \end{array}\begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{array}$$

■

Theorem 0.11.17 Let B and A be, respectively, the circuit matrix and incidence matrix of a self-loop-free digraph such that the columns in B and A are arranged using the same order of edges. Then $A \cdot B^T = B \cdot A^T = O$, where superscript T denotes the transposed matrix.

Proof. Consider the m^{th} row in B and the k^{th} row in A. If the circuit m does not include any edge incident on vertex k, the product of the two rows is clearly zero. On the other hand, if vertex k is in circuit m, there are exactly two edges (say x and y) incident on k that are also in circuit m. This situation can occur in only four different ways. (The other four cases with the orientation of m reversed are identical to these when x and y are interchanged.)



The possible entries in row k of A and row m of B in column positions x and y are tabulated for each of these four cases.

Case	Row k		Row m		Dot Product
	Column x	*Column y*	*Column x*	*Column y*	*Row k·Row m*
(a)	−1	1	1	1	0
(b)	1	−1	−1	−1	0
(c)	−1	−1	1	−1	0
(d)	1	1	−1	1	0

In each case, the dot product is zero. Hence the theorem. ■

Theorem 0.11.18 (Binet-Cauchy theorem) If Q and R are k by m and m by k matrices respectively, with $k < m$, then the determinant of the product i.e.,
$\det(QR)$ = the sum of the products of corresponding major determinants of Q and R.

Example:

$$Q = \begin{bmatrix} 4 & -3 & -2 \\ 2 & -1 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 3 & -2 \end{bmatrix};$$

$$\begin{aligned}\det(QR) &= \det\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}\cdot\det\begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix} + \det\begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix} \\ &\quad \cdot\det\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} + \det\begin{bmatrix} -3 & -2 \\ -1 & 0 \end{bmatrix}\cdot\det\begin{bmatrix} -2 & 0 \\ 3 & -2 \end{bmatrix} \\ &= 2\cdot(-2) + 4\cdot 1 + (-2)\cdot 4 = -8.\end{aligned}$$

Theorem 0.11.19 Let A_f be the reduced incidence matrix of a connected digraph. Then the number of spanning trees in the graph equals the value of the $\det(A_f \cdot A_f^T)$.

Proof. According to the Binet-Cauchy theorem,
$\det(A_f \cdot A_f^T)$ = sum of the products of all corresponding majors of A_f and A_f^T.
Every major of A_f or A_f^T is zero unless it corresponds to a spanning tree, in which case its value is 1 or −1. Since both majors of A_f and A_f^T have the same value +1 or −1, the product is +1 for each spanning tree. ■

Definition 0.11.27 — ADJACENCY MATRIX OF A DIGRAPH. Let G be a digraph with n vertices, containing no parallel edges. Then the adjacency matrix $X = [x_{ij}]$ of the digraph G is an n by n $(0,1)$-matrix defined as

$$x_{ij} = \begin{cases} 1, & \text{if there is an edge directed from } i^{th} \text{ vertex to } j^{th} \text{vertex,} \\ 0, & \text{otherwise.} \end{cases}$$

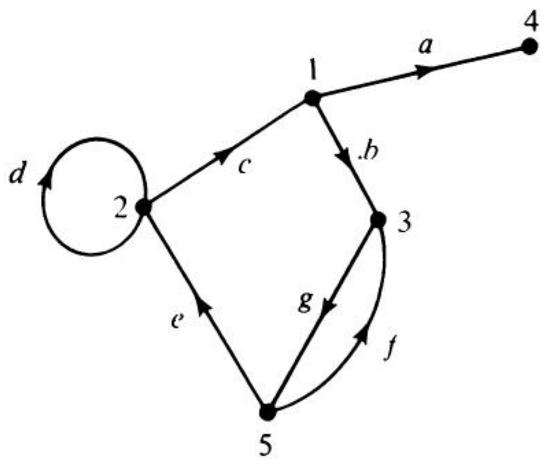
R X is a symmetric matrix if and only if G is a symmetric digraph.

R Every non-zero element on the main diagonal represents a self-loop at the corresponding vertex.

R There is no way of showing parallel edges in X. This is why the adjacency matrix is defined only for a digraph without parallel edges.

R The sum of each row equals the out-degree of the corresponding vertex, and the sum of each column equals the in-degree of the corresponding vertex. The number of non-zero entries in X equals the number of edges in G.

■ **Example 0.26** Write the adjacency matrix for the following digraph:



Solution:

$$X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

■

0.11.2 ENUMERATION OF GRAPHS

Enumeration means counting or numbering.
All graph-enumeration problems tackle into two types:

1. Counting the number of different graphs (or digraphs) of a particular kind, e.g., all connected, simple graphs with eight vertices and two circuits.
2. Counting the number of subgraphs of a particular type in a given graph G, such as the number of edge-disjoint paths of length k between vertices a and b in G. e.g., a matrix representation of graph G and manipulations of this matrix.

R $\binom{n}{r} = {}^nC_r = \dfrac{n!}{(n-r)!\,r!}$ and ${}^nC_r = {}^nC_{n-r}$

Theorem 0.11.20 The number of simple, labelled graphs of n vertices is $2^{\frac{n(n-1)}{2}}$.

Proof. The numbers of simple graphs of n vertices and $0, 1, 2, \ldots, \dfrac{n(n-1)}{2}$ edges are obtained by substituting $0, 1, 2, \ldots, \dfrac{n(n-1)}{2}$ for e in expression $\binom{\frac{n(n-1)}{2}}{e}$. The sum of all such numbers is the number of all simple graphs with n vertices. Then the use of the following identity proves the theorem:

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k-1} + \binom{k}{k} = 2^k$$

■

2nd Proof:

Proof. The maximum number of edges possible in a simple graph with n vertices is $\binom{n}{2}$ where $\binom{n}{2} = \dfrac{n(n-1)}{2}$.
$\therefore$ The number of simple graphs possible with n vertices = $2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}}$. ■

R The maximum number of simple, labelled graphs possible with n vertices and e edges is $\binom{\binom{n}{2}}{e}$.

■ **Example 0.27** Find the number of simple labelled graphs with 5 vertices and 6 edges.
Solution: Here $n = 5$ and $e = 6$.

Then $\binom{n}{2} = \binom{5}{2} = \dfrac{5 \times \not{4}^{\,2}}{\not{2}_{\,1}} = 10.$

The number of simple, labelled graphs possible with n vertices and e edges$= \binom{\binom{n}{2}}{e} = \binom{10}{6} =$

$\binom{10}{4} = \dfrac{10 \times \not{9}^{\,3} \times \not{8} \times 7}{\not{4} \times \not{3} \times \not{2} \times 1} = 210 \qquad \because \binom{n}{r} = \binom{n}{n-r}.$ ■

0.11.3 COUNTING LABELLED TREES

Theorem 0.11.21 There are n^{n-2} labelled trees with n vertices (where $n \geq 2$).

Proof. Let the n vertices of a tree T be labelled as $1, 2, 3, \ldots.., n$. Remove the pendant vertex (and the edge incident on it) having the smallest label, which is a_1, (say). Suppose that b_1 was the vertex adjacent to a_1. Among the remaining $n-1$ vertices, let a_2 be the pendant vertex with the smallest label, and b_2 be the vertex adjacent to a_2. Remove the edge (a_2, b_2). This operation is repeated on the remaining $n-2$ vertices, and then on $n-3$ vertices and so on. The process is terminated after $n-2$ steps, when only two vertices are left. The tree T defines the unique sequence

$$(b_1, b_2, b_3, \ldots\ldots., b_{n-2}) \qquad \longrightarrow (1)$$

Conversely, given a sequence (1) of $n-2$ labels, an n-vertex tree can be constructed uniquely, as follows: Determine the first number in the sequence

$$1, 2, 3, \ldots, n \qquad \longrightarrow (2)$$

that does not appear in sequence (1). This number is a_1. And thus the edge (a_1, b_1) is defined. Remove b_1 from sequence (1) and a_1 from (2). In the remaining sequence of (2) find the first number that does not appear in the remainder of (1). This would be a_2, and thus the edge (a_2, b_2) is defined. The construction is continued till the sequence (1) has no element left. Finally, the last two vertices remaining in (2) are joined.

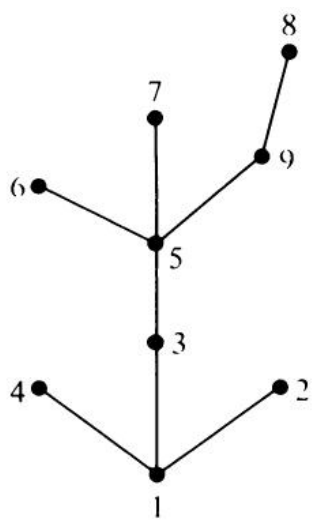
For each of the $(n-2)$ elements in sequence (1) we can select any one of n numbers, thus forming

$$n^{n-2} \qquad \longrightarrow (3)$$

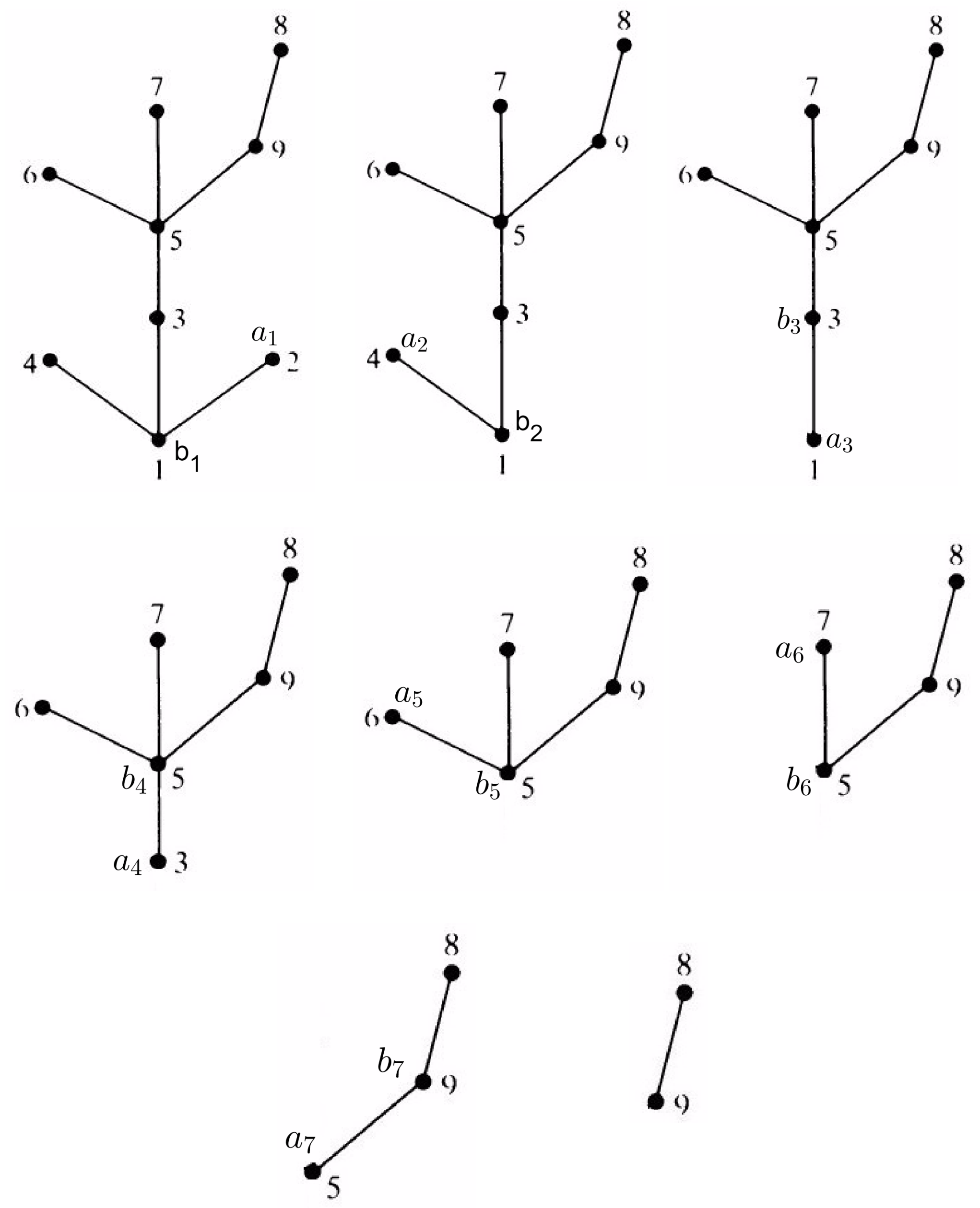
$(n-2)$-tuples, each defining a distinct labelled tree of n vertices. And since each tree defines one of these sequences uniquely, there is a one-to-one correspondence between the trees and the n^{n-2} sequences. Hence the theorem. ■

(R) A vertex i appears in sequence $(b_1, b_2, b_3, \ldots\ldots., b_{n-2})$ if and only if it is not pendant vertex.

■ **Example 0.28** Construct a sequence for the following tree:



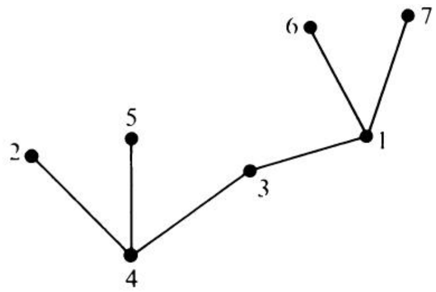
Solution:



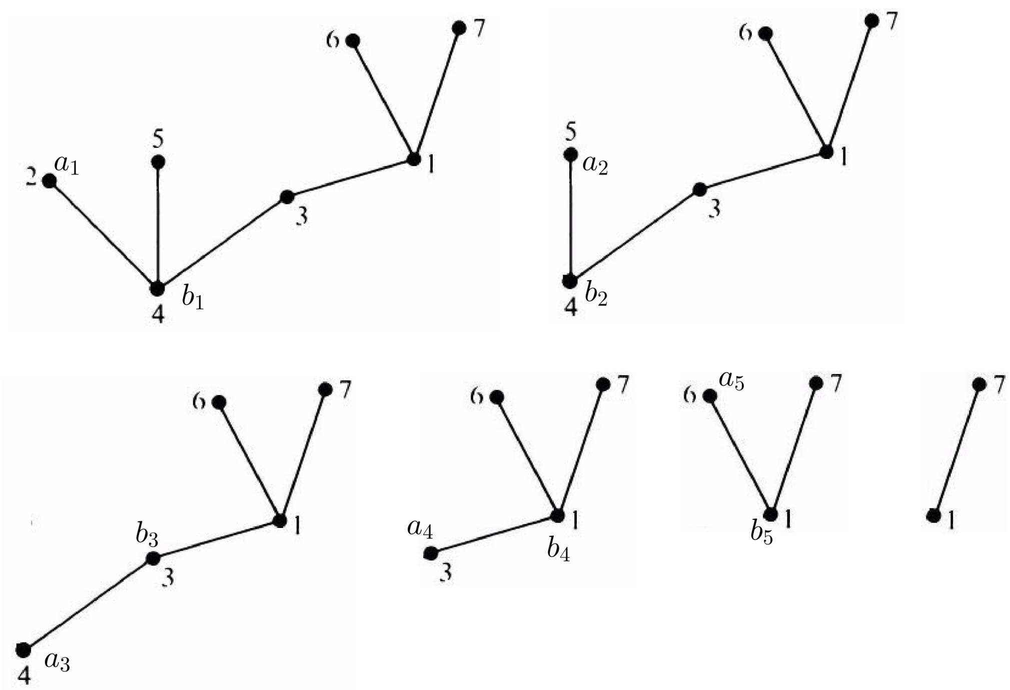
Remove the pendant vertex having the smallest label, which is 2 $(= a_1)$.
1 $(= b_1)$ is the vertex adjacent to $(= a_1)$. Remove a_1
Among the remaining $n - 1 = 9 - 1 = 8$ vertices, 4 $(= a_2)$ is the pendant vertex with the smallest label and 1 $(= b_2)$ is the vertex adjacent to (a_2). Remove a_2.
Among the remaining $n - 2 = 9 - 2 = 7$ vertices, 1 $(= a_3)$ is the pendant vertex with the smallest label and 3 $(= b_3)$ is the vertex adjacent to (a_3). Remove a_3.
Among the remaining $n - 3 = 9 - 3 = 6$ vertices, 3 $(= a_4)$ is the pendant vertex with the smallest label and 5 $(= b_4)$ is the vertex adjacent to (a_4). Remove a_4.
Among the remaining $n - 4 = 9 - 4 = 5$ vertices, 6 $(= a_5)$ is the pendant vertex with the smallest label and 5 $(= b_5)$ is the vertex adjacent to (a_5). Remove a_5.
Among the remaining $n - 5 = 9 - 5 = 4$ vertices, 7 $(= a_6)$ is the pendant vertex with the smallest label and 5 $(= b_6)$ is the vertex adjacent to (a_6). Remove a_6.

Among the remaining $n-6=9-6=3$ vertices, 5 $(=a_7)$ is the pendant vertex with the smallest label and 9 $(=b_7)$ is the vertex adjacent to (a_7). Remove a_7.
Now, only one edge is left. Stop the process.
Hence, the required sequence is $(1,1,3,5,5,5,9)$. ■

■ **Example 0.29** Construct a sequence for the following tree:

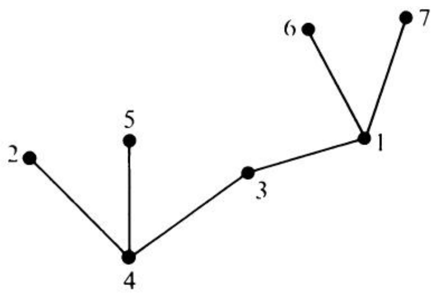


Solution:



Remove the pendant vertex having the smallest label, which is 2 $(=a_1)$.
4 $(=b_1)$ is the vertex adjacent to $(=a_1)$. Remove a_1
Among the remaining $n-1=7-1=6$ vertices, 5 $(=a_2)$ is the pendant vertex with the smallest label and 4 $(=b_2)$ is the vertex adjacent to (a_2). Remove a_2.
Among the remaining $n-2=7-2=5$ vertices, 4 $(=a_3)$ is the pendant vertex with the smallest label and 3 $(=b_3)$ is the vertex adjacent to (a_3). Remove a_3.
Among the remaining $n-3=7-3=4$ vertices, 3 $(=a_4)$ is the pendant vertex with the smallest label and 1 $(=b_4)$ is the vertex adjacent to (a_4). Remove a_4.
Among the remaining $n-4=7-4=3$ vertices, 6 $(=a_5)$ is the pendant vertex with the smallest label and 1 $(=b_5)$ is the vertex adjacent to (a_5). Remove a_5.
Now, only one edge is left. Stop the process.
Hence, the required sequence is $(4,4,3,1,1)$. ■

■ **Example 0.30** Construct a seven-vertex tree for the sequence $(4,4,3,1,1)$.
Solution:
Given sequence is $(4,4,3,1,1)$ $\longrightarrow (1)$.
Find the first number in the sequence $1,2,3,4,5,6,7$ $\longrightarrow (2)$,
that does not appear in sequence (1). This number clearly is 2. And thus the edge $(2,4)$ is defined. Remove 4 from sequence (1) and 2 from (1). In the remaining sequence of (2) find the first number that does not appear in the remainder of (1). This would be 5, and thus the edge $(5,4)$ is defined. The construction is continued till the sequence (1) has no element left. Finally, the last two vertices remaining in (2) are joined.
we can construct a seven-vertex tree as follows: $(2,4)$ is the first edge. The second is $(5,4)$. Next, $(4,3)$. Then $(3,1)$, $(6,1)$, and finally $(7,1)$.

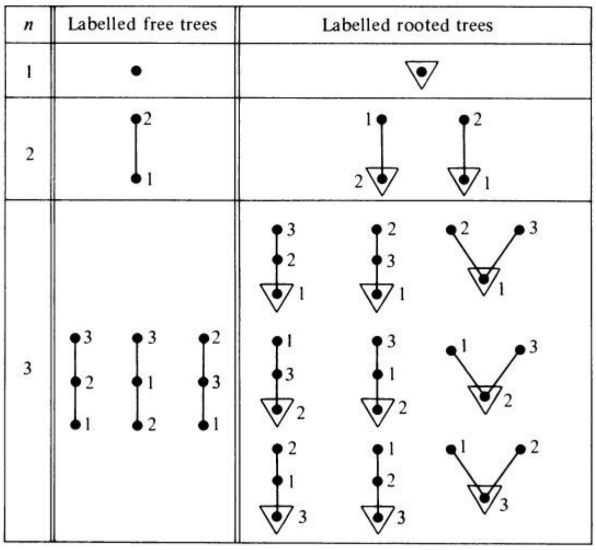


■

■ **Example 0.31** Construct a nine-vertex tree for the sequence $(1,1,3,5,5,5,9)$.
Solution:
Please try yourself. ■

Theorem 0.11.22 The number of different rooted, labelled trees with n vertices is n^{n-1}.

Examples: Rooted labelled trees of one, two and three vertices:



Question Papers

Reg. No. ☐☐☐☐☐☐☐☐☐

BSCMTC 359

Credit Based VI Semester B.Sc. Degree Examination, April/May 2017
(Semester Scheme) (2016 – 17 Batch Onwards)
MATHEMATICS
Graph Theory (Special Paper 8a)

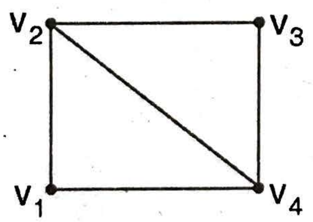
Time : 3 Hours Max. Marks : 120

Instructions: *1) Answer **any ten** questions from (Part – **A**). **Each** question carries **3** marks.*
*2) Answer **five full** questions from (Part – **B**) choosing **one full** question from **each** Unit.*
*3) Scientific calculators are **allowed**.*

PART – A **(10×3=30)**

1. If a graph has exactly two vertices of odd degree, then prove that there must be a path joining these two vertices.

2. Define a spanning tree and draw any two spanning trees of the following graph, which are not isomorphic.



3. Define the terms :

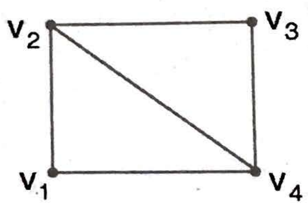
   a) Euler graph

   b) Complete graph.

4. Prove that the vertex connectivity of any graph G can never exceed the edge connectivity of G.

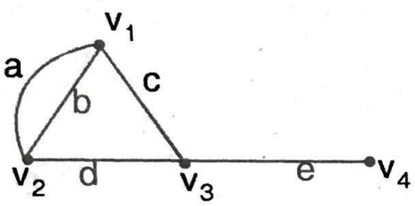
5. Define planar and non-planar graphs and give an example of a non-planar graph.

6. Draw the geometric dual of the graph below.



7. Define reduced incidence matrix and prove that the reduced incidence matrix of a tree is non-singular.

8. Write the path matrix $P_{(v_1, v_4)}$ for the vertices v_1 and v_4 in the following graph.



9. Define cut set matrix of a graph G. How are the parallel edges in G represented in the cut set matrix ?

10. Define :

   a) Proper colouring of a graph.

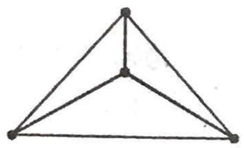
   b) Chromatic number.

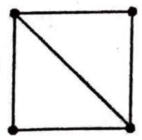
11. Write the chromatic polynomial of :

   i) a complete graph with 4 vertices

   ii) a tree with 15 vertices.

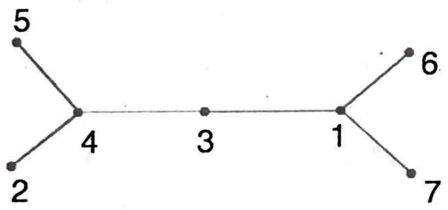
12. Write the chromatic number of the following graphs :

a) 

b) 

13. Define strongly connected and weakly connected digraphs with examples.

14. Find the sequence associated with the 7-vertex tree labelled below.



15. Find the number of simple labelled graphs with 5 vertices and 6 edges.

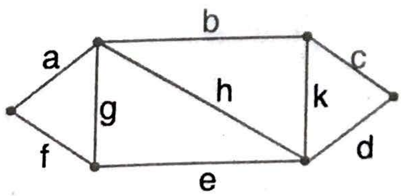
PART – B

Unit – I

1. a) Prove that a simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges. **6**

b) Prove that a graph is a tree if and only if it is minimally connected. **6**

c) Define a fundamental circuit and list all the fundamental circuits with respect to the spanning tree T = {a, g, b, k, c} of the following graph. **6**



2. a) Prove that a connected graph G is an Euler graph if and only if all the vertices of G are of even degree. **6**

b) i) Prove that the number of vertices in a binary tree is odd.

ii) Prove that any connected graph with n vertices and (n – 1) edges is a tree. **6**

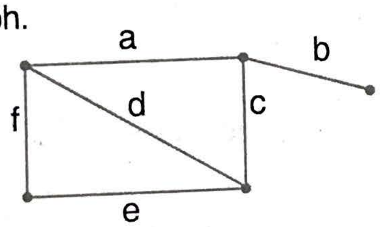
c) Define distance between two vertices in a graph and show that distance between vertices in a connected graph is a metric. **6**

Unit – II

3. a) With respect to a given spanning tree T, prove that a chord c that determines a fundamental circuit Γ occurs in every fundamental cut set associated with the branches in Γ, and in no other. **6**

b) Prove that Kuratowski's second graph $K_{3,3}$ is non-planar. **6**

c) Define a cut set in a connected graph G and list all the cut sets in the following graph. **6**



4. a) Prove that in a connected planar graph with n vertices, e edges, there are e – n + 2 regions. **6**

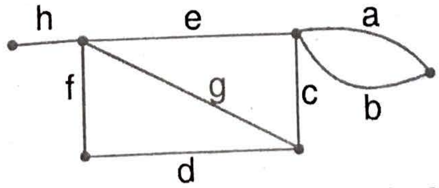
b) Prove that a graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane. **6**

c) Prove that in a graph every circuit has an even number of edges in common with any cut set. **6**

Unit – III

5. a) Prove that the rank of the incidence matrix of a connected graph of n vertices is n – 1. **6**

b) Define a circuit matrix of a graph and write the circuit matrix of the following graph : **6**

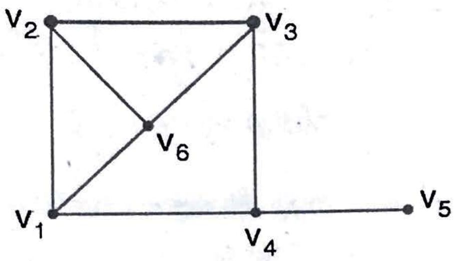


c) Prove that the rank of the cut set matrix C (G) of a graph G of n vertices and e edges is equal to the rank of incidence matrix A(G). **6**

6\. a) Prove that the ring sum of two circuits in a graph G is either a circuit or an edge disjoint union of circuits. **6**

b) Let B and A be respectively the circuit matrix and the incidence matrix of a self loop free graph whose columns are arranged in the same order of edges. Then prove that $A \cdot B^T \equiv 0 \pmod 2$. **6**

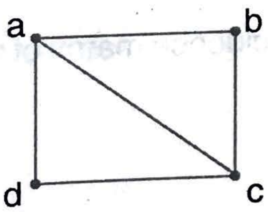
c) Write the adjacency matrix of the following graph. **6**



Unit – IV

7\. a) Prove that every tree with two or more vertices is 2 – chromatic. **6**

b) Write the chromatic polynomial of the following graph with explanation. **6**

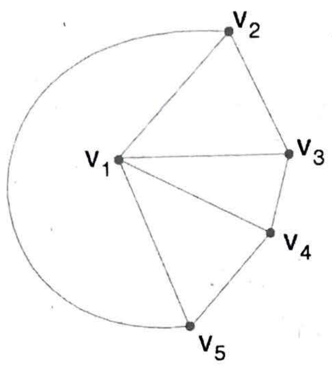


c) Prove that a graph with at least one edge is 2 – chromatic if and only if it has no circuits of odd length. **6**

8. a) Prove that a graph on n vertices is a complete graph if and only if its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$. **6**

b) Prove that for a tree with n vertices, the chromatic polynomial is $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$. **6**

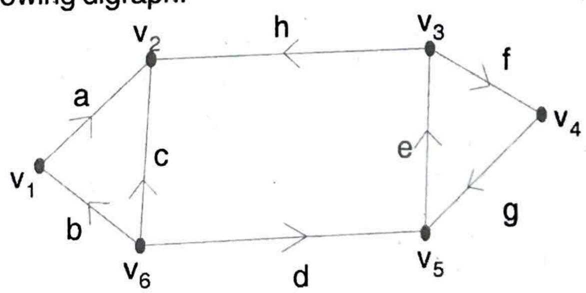
c) Find the chromatic polynomial of the graph given below. **6**



Unit – V

9. a) Show that the determinant of every square submatrix of the incidence matrix of a diagraph is 1, –1 or 0. **6**

b) Define incidence matrix of a digraph and write the incidence matrix of the following digraph. **6**



c) Let A_f be the reduced incidence matrix of a connected digraph. Then prove that the number of spanning trees in the graph equals the value of $\det(A_f \cdot A_f^T)$. **6**

10\. a) Prove that there are n^{n-2} labelled trees with n vertices, $n \geq 2$. 6

b) If $X = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$ is the adjacency matrix of a digraph. 6

i) Draw the corresponding digraph

ii) Write the out-degree of v_4

iii) Write the in-degree of v_2.

c) Let B and A be respectively the circuit matrix and the incidence matrix of a self loop free digraph such that the columns in B and A are arranged in the same order of edges. Then prove that $A \cdot B^T = B \cdot A^T = 0$. 6

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al No. of Printed Pages : 7
al No. of Questions : 25

BSCMTC359

No. : 1218

redit Based VI Semester B.Sc. Degree Examination, May 2018

BSCMTC 359 : MATHEMATICS

(2016-17 Batch Onwards) Paper - VIII (a)

Graph Theory

me : 3 Hours

Max. Marks : 120

structions : 1) Answer any ten questions from Part A. Each question carries 3 marks.

2) Answer to Part A should be written in the first few pages of the answer book before answers to Part B.

3) Answer five full questions from Part B choosing one full question from each Unit.

4) Scientific calculators are allowed.

Part - A

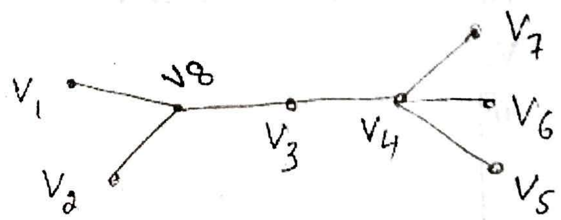
[10 x 3 = 30]

) Define

a) Hamiltonian Circuit.

b) Fundamental circuit in a connected graph G.

) Find the center of the graph by finding eccentricity of each vertex in the following graph.



) If a graph has exactly two vertices of odd degree, then prove that there must be a path joining these two vertices.

P.T.O.

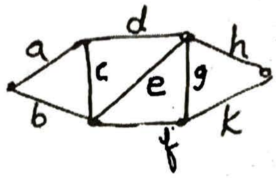
Q4) Define

a) Separable Graph

b) Planar Graph

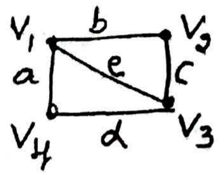
Q5) Prove that the vertex connectivity of any graph G can never exceed the edge connectivity of 6.

Q6) List any three fundamental cutsets of the following graph with respect to the spanning tree {b, e, c, h, k}.



Q7) Prove that the reduced incidence matrix of a tree is nonsingular.

Q8) Write the path matrix P (V_1, V_3) for the following graph.



Q9) Draw the graph whose adjacent matrix is given below.

	V_1	V_2	V_3	V_4
V_1	1	0	1	0
V_2	0	0	1	1
V_3	1	1	0	0
V_4	0	1	0	0

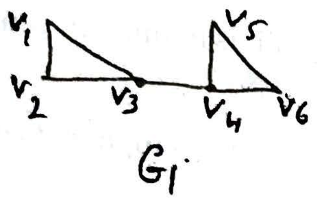
Q10) Define

a) Proper colouring

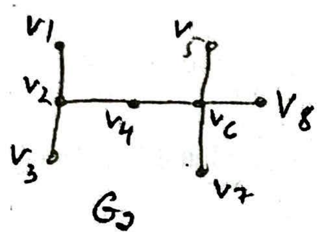
b) Chromatic number of a graph

Q11) Write the chromatic number for the following graphs.

a)



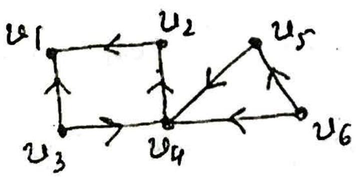
b)



Q12) What is the chromatic polynomial of a tree with 10 vertices?

Q13) Define asymmetric digraph with an example.

Q14) Define balanced digraph and write the in degree and out degree of each vertex in the following graph.



Q15) Draw the labelled tree with respect to the sequence [1, 4, 11].

Part - B

UNIT - I

Q16) a) Define Euler Graph. Prove that a connected graph G is an Euler graph if and only if all vertices are of even degree. [6]

b) Define a spanning tree. Prove that a graph G is a tree if and only if there is one and only one path between every pair of vertices. [6]

c) Prove that a graph G is a tree if and only if it is minimally connected. [6]

Q17) a) Prove that a simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges. [6]

b) Define distance in a graph. Prove that distance between two vertices in a connected graph is a metric. [6]

c) Define Pendent vertex. Prove that there are at least two pendant vertices in a tree. [6]

UNIT - II

Q18) a) Prove that a connected planar graph with n vertices and e edges has e–n+2 regions. [6]

b) Prove that a complete graph of five vertices is nonplanar. [6]

c) Prove that a graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane. [6]

Q19) a) Prove that every cutset has even number of edges in common with any circuit. [6]

b) Define vertex connectivity and edge connectivity Using Eulers formula prove that K_{33} is nonplanar. [6]

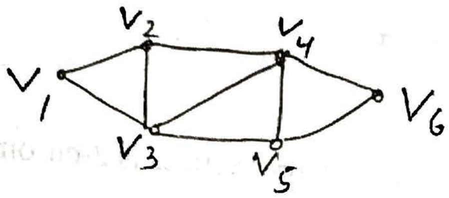
c) Prove that with respect to a given spanning tree T, a branch b_i that determines a fundamental. Cutset S is contained in every fundamental circuit associated writh the chords in S and in no other. [6]

UNIT - III

Q20) a) If B is a circuit matrix of a connected graph G with n vertices and e edges, prove that rank of B = e–n+1. **[6]**

b) Show that the rank of a cutset matrix C (G) = rank of the incidence matrix A (G) **[6]**

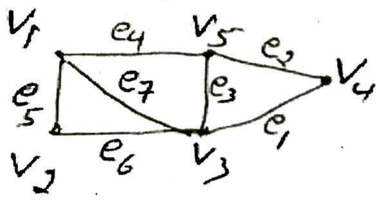
c) Write the adjacency matrix of the following graph. **[6]**



Q21) a) If A and B are incidence and circuit matrix of a graph G without self loops, whose columns are arranged using same order of edges, then prove that $AB^T = A^TB \equiv O \pmod 2$. **[6]**

b) Prove that the ring sum of two circuits in a graph G is either a circuit or an edge disjoint union of circuits. **[6]**

c) Write the fundamental circuit matrix of the following graph with respect to the spanning tree $\{e_1\ e_2\ e_4\ e_5\}$ **[6]**

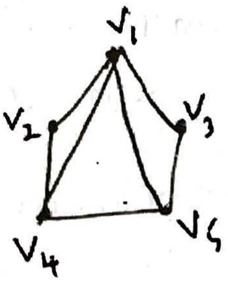


UNIT - IV

Q22) a) Prove that a graph with at least one edge is two chromatic if and only if it has no circuits of odd length. **[6]**

b) Prove that a graph with n vertices is a complete graph if and only if its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots\ldots\ldots\ldots (\lambda - n + 1)$ [6]

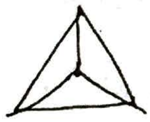
c) Write the chromatic polynomial of the following graph with explanation. [6]

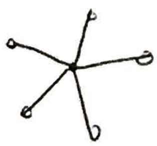


Q23) a) Prove that every tree with two or more vertices is 2-chromatic. [6]

b) Prove that for a tree with n vertices, chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$. [6]

c) Find the chromatic polynomial of the graph. [6]

i) 

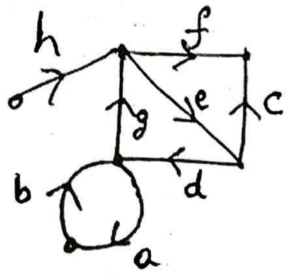
ii) 

UNIT - V

Q24) a) Prove that the determinant of every square submatrix of the incidence matrix of a digraph is –1, 1, or 0. [6]

b) Prove that there are n^{n-2} labelled trees with n vertices, $n \geq 2$. [6]

c) Define a circuit matrix of a digraph. Write the circuit matrix of the following digraph. **[6]**



Q25) a) Let B and A be respectively the circuit matrix and the incidence matrix of a self loop free digraph such that the columns in B and A are arranged in the same order of edges. Then prove that $A.\ B^T = B.A^T = 0$. **[6]**

b) Let A_f be the reduced incidence matrix of a connected digraph. Then prove that the number of spanning trees in the graph equals the value of $\det(A_f \cdot A_f^T)$. **[6]**

c) Define incidence matrix of a digraph. Draw the digraph for following incidence matrix. **[6]**

	a	b	c	d	e	f	g	h
V_1	0	0	0	−1	0	1	0	0
V_2	0	0	0	0	1	−1	1	−1
V_3	0	0	0	0	0	0	0	1
V_4	−1	−1	−1	0	−1	0	0	0
V_5	0	0	1	1	0	0	−1	0
V_6	1	1	0	0	0	0	0	0

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Reg. No. | | | | | | | | | |

BSCMTC 359

VI Semester B.Sc. Degree Examination, April/May 2019

(Credit Based Semester Scheme)

(2016-17 Batch onwards)

MATHEMATICS

Paper VIII(a) – Graph Theory

Time : 3 Hours] [Max. Marks : 120

Note : *A single answer booklet containing 40 pages will be issued, No additional sheets will be issued.*

Instructions : 1. *Answer any ten questions from Part A. Each question carries 3 marks.*

2. *Answer to Part A should be written in the first few pages of the main answer book, before Part B.*

3. *Answer five full questions from Part B choosing One full question from each Unit.*

4. *Scientific calculators are allowed.*

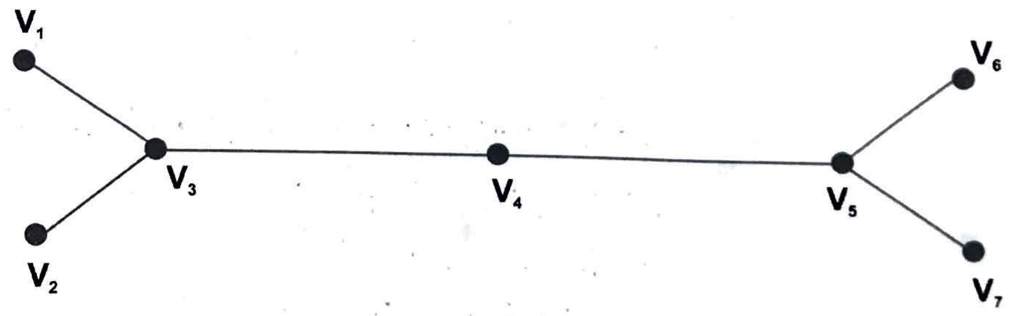
PART – A

Answer **any ten** questions : **(10 × 3 = 30)**

1. In a binary tree T of 'n' vertices, prove that the number of pendant vertices is $p = \dfrac{n+1}{2}$.

2. Prove that the number of vertices having odd degree in a graph is even.

3. Find the centre of the graph given below by finding eccentricity of each vertex :



P.T.O.

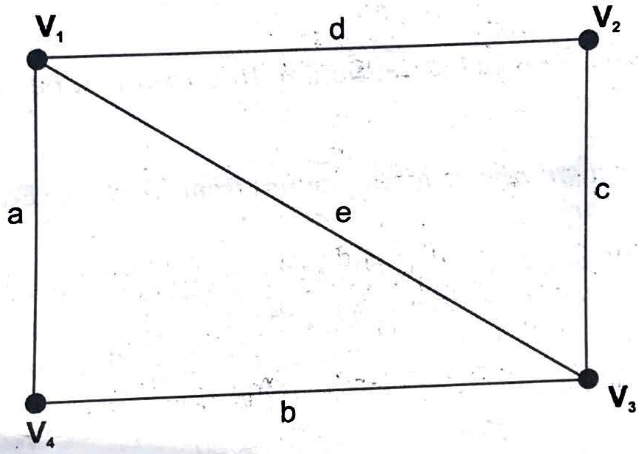
4. Define (a) Vertex connectivity of a graph (b) Fundamental cutset in a graph.

5. If any simple planar graph with f regions, n vertices and e edges, prove that $e \leq 3n - 6$.

6. Prove that edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

7. Define cutset matrix of a connected graph G.

8. Write the path matrix $P(v_1, v_3)$ of the following graph :

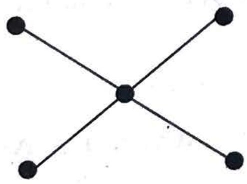


9. Prove that the reduced incidence matrix of a tree is non-singular.

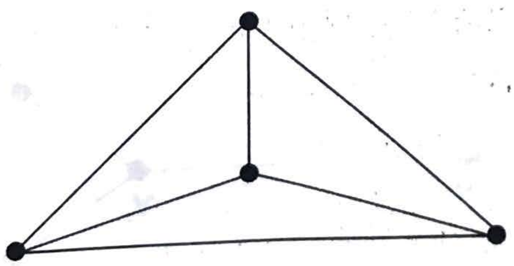
10. Define : (a) Proper colouring of a graph (b) Chromatic number of a graph.

11. Write the chromatic polynomial of the following graphs :

(a)

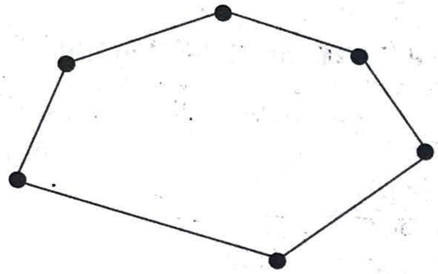


(b)

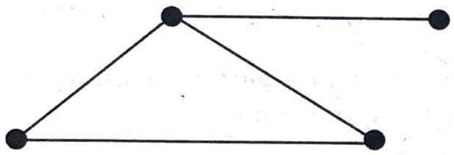


12. Write the chromatic number of the following graphs :

(a)

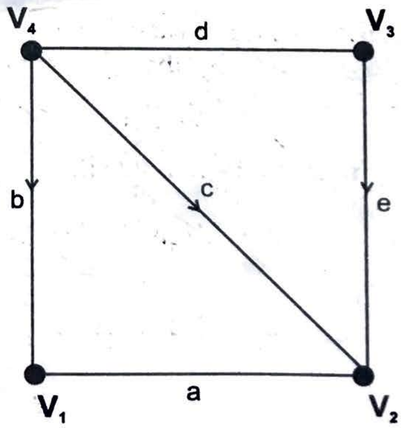


(b)



13. Prove that the number of simple labelled graphs of n vertices is $2^{\frac{n(n-1)}{2}}$.

14. Draw the circuit matrix of the following digraph :



15. Define the terms :

(a) Balanced digraph

(b) Symmetric digraph

PART – B

UNIT I

16. (a) Prove that a given connected graph G is an Euler graph if and only if all vertices of G are of even degree.

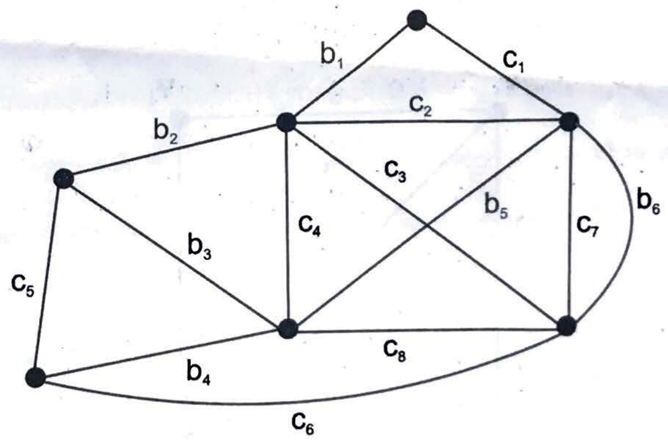
(b) If G is a simple graph with 'n' vertices and 'k' components, prove that G can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

(c) Define a spanning tree and show that every connected graph has at least one spanning tree. **(6 + 6 + 6)**

17. (a) Define distance between two vertices in a graph and show that distance between vertices in a connected graph is a metric.

(b) Prove that a tree with n vertices has $n-1$ edges.

(c) Define a fundamental circuit and list any five fundamentals circuits of the following graph with respect to a spanning tree $\{b_1, b_2, b_3, b_4, b_5, b_6\}$: **(6 + 6 + 6)**



UNIT II

18. (a) Prove that Kuratowski's second graph $K_{3,3}$ is non-planar.

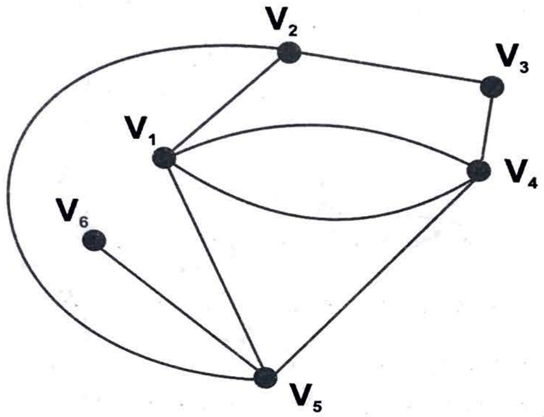
(b) Prove that a connected planar graph G with n vertices, e edges, there are $e-n+2$ regions.

(c) Prove that every circuit has an even number of edges in common with any cutset. **(6 + 6 + 6)**

19. (a) Prove that a graph can be embedded in the surface of a sphere if and only if it can be embedded in a plane.

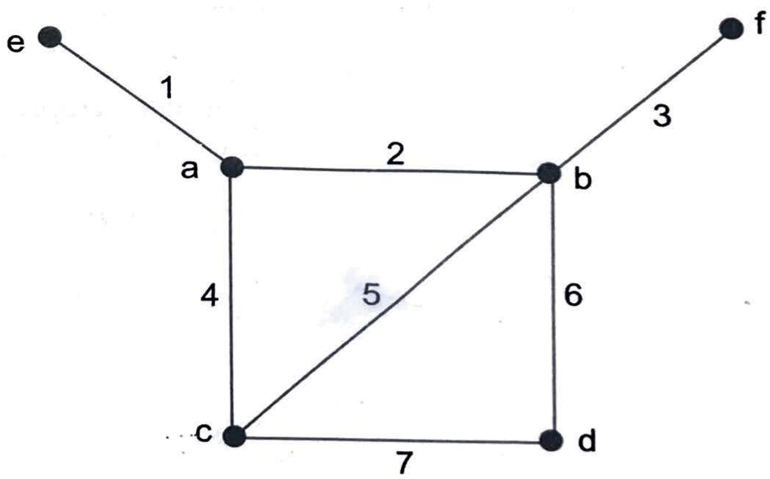
(b) Prove that with respect to any spanning tree T, a chord C_i that determines a fundamental circuit Γ occurs in each fundamental cutset associated with the branches and in no other.

(c) Draw a geometric dual of the following graph : **(6 + 6 + 6)**



UNIT III

20. (a) Write the adjacency matrix of the following graph :

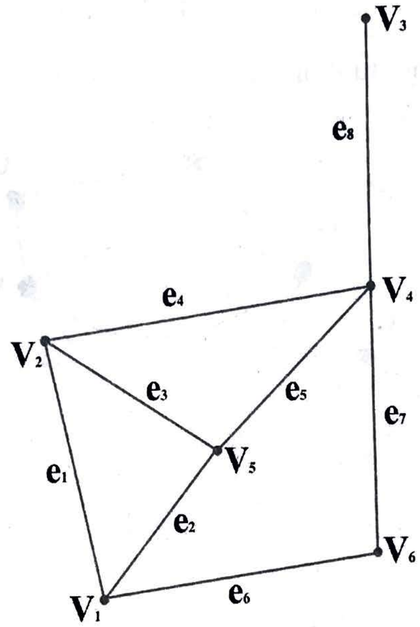


(b) Prove that the ring sum of two circuits in a graph is either a circuit or an edge disjoint union of circuits.

(c) If B is a circuit matrix of a connected graph G with n vertices and e edges, then prove rank of $B = e - n + 1$. **(6 + 6 + 6)**

21. (a) Let B and A be respectively the circuit matrix and the incidence matrix of a self-loop free graph whose columns are arranged using the same order of edges. Prove that every row of A is orthogonal to every row of B, that is $A \cdot B^T \equiv 0 (\bmod 2)$.

(b) Write the incidence matrix of the following graph :

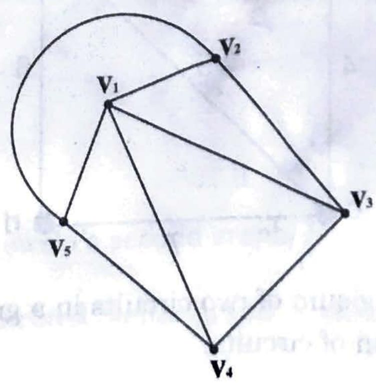


(c) Prove that ring sum of any two circuits in a graph G is either a circuit or an edge disjoint union of circuits. **(6 + 6 + 6)**

UNIT IV

22. (a) Prove that a graph on n vertices is a complete graph if and only if its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1)\dots(\lambda - n + 1)$.

(b) Find the chromatic polynomial of the following graph :



(c) Prove that a graph with at least one edge is two chromatic if and only if it has no circuits of odd length. **(6 + 6 + 6)**

23. (a) Prove that every tree with two or more vertices is two chromatic.

(b) Let a and b be two non-adjacent vertices in a graph G. Let G' be a graph obtained by adding an edge between a and b and G'' be a simple graph obtained from G by fusing the vertices a and b together and replacing sets of parallel edges with a single edge. Prove that $P_n(\lambda)$ of $G = P_n(\lambda)$ of $G' + P_{n-1}(\lambda)$ of G''.

(c) Prove that a graph of 'n' vertices is a tree if and only if its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$. **(6 + 6 + 6)**

UNIT V

24. (a) Prove that the determinant of every square submatrix of the incidence matrix of a digraph is -1, 1 or 0.

(b) Prove that there are n^{n-2} labelled trees with n vertices $n \geq 2$.

(c) Define an incidence matrix of a digraph and draw the digraph for the incidence matrix given below :

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix} \end{array}$$

(6 + 6 + 6)

25. (a) Let B and A be respectively the circuit matrix and the incidence matrix of a self-loop tree digraph such that columns in B and A are arranged in the same order of edges. Then prove that $A \cdot B^T = B \cdot A^T = 0$.

(b) Let A_f be the reduced incidence matrix of a connected digraph. Prove that the number of spanning trees in a graph equals the value of the $\det(A_f \cdot A_f^T)$.

(c) Construct a 9 vertex labelled tree which yields the sequence.

(1, 1, 3, 5, 5, 5, 9) **(6 + 6 + 6)**

Bibliography

Books

[1] GRAPH THEORY with Applications to Engineering and Computer Science, NARSINGH DEO, PRENTICE-HALL, INC.

[2] Graph Theory, Frank Harary, Addison-Wesley Publishing Company.

[3] College Graph Theory, V. R. Kulli, Vishwa International Publications.

[4] Introduction to Graph Theory, Douglas B. West, Pearson.

[5] A Textbook of Graph Theory, Balakrishnan, R., Ranganathan, K., Springer.