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Fully Solved

Analytical Geometry 3D

(Co-ordinate Solid Geometry)

(For Degree and Honours Students of Indian Universities & for Various Competitive Examinations like P.C.S.& I.A.S. etc.)

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Systems of Co-ordinates

§ 1. **Introduction.** Students know well that in *co-ordinate geometry of two dimensions* (*i.e. plane analytical geometry*) the position of a point in a plane is referred to two intersecting lines (in the plane of the point) called **the axes of reference** and their point of intersection called **the origion of co-ordinates.** The axes are called **rectangular axes** if they are at right angles, otherwise they are called **oblique axes.** Whatever the axes may be, they divide the plane into four quadrants called the first, second, third and fourth quadrants respectively.

But it is not always possible to determine the positions of all the points we can imagine with reference to above co-ordinate axes. For example, codsider the five corners of a rectangular parallelopiped, they do not lie in one plane. Such points are called *points in space.* A point in space can be **demonstrated** as follows :

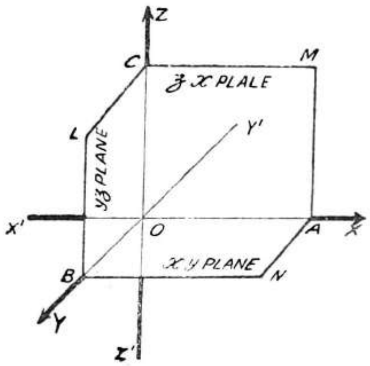
Consider your study room and let dimensions of the room be that of a rectangular parallelopiped. Now consider any particle in air, then this particle in air is a point in space.

The geometry of such points in space is discussed in **"Analytical geometry of three dimensions"** also called **"Solid geometry."**

§ 2. **Definitions. Origin, axes and co-ordinate planes.**

Draw two mutually perpendicular lines $X'OX$ and $Z'OZ$ in the plane of the paper. Let these lines intersect at O, then through O imagine a third line $Y'OY$ perpendicular to both of the above lines, so that OY is perpendicular to the plane of the paper and is directed upwards.

The point O is called the **origin.** The three mutually perpendicular lines namely $X'OX$, $Y'OY$ and $Z'OZ$ are called **the axes of reference** (rectangular), and are said to be x-axis, y-axis and z-axis respectively. OX is taken to be positive direction of x-axis whereas OX' as negative direction of x-axis. In a similar way OY and OZ are taken to be the +ve directions and OY' and OZ' as −ve directions of y and z-axes respectively.

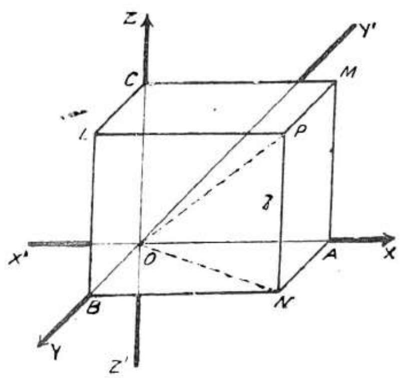


When these three axes are taken in pairs, they give us three planes YOZ, ZOX and XOY. These three planes are called yz, zx and xy-planes respectively. The set of these three planes is called the **set of co-ordinate planes** (rectangular).

The axes are **oblique axes**, if they are not rectangular.

Note. In the rest of the book the axes will be assumed to be rectangular unless otherwise stated.

§ 3. Co-ordinates of a point in space. Consider a point P in space. Through P draw a plane $PNAM$ parallel to YOZ plane *i.e.* perpendicular to x-axis meeting it in the point A; if $OA=a$, then a is called the x-co-ordinate of P. Similarly through P draw planes $PNBL$ parallel to the plane ZOX and $PMCL$ parallel to the plane XOY meeting y and z axes in the points B and C respectively; if $OB=b$ and $OC=c$ then b is called y-co-ordinate of P and c is called z-co-ordinate of P. The three numbers a, b, c are called the co-ordinates of the point P and are written by



ordered triads of the form (a, b, c). These co-ordinates are measured positive or negative in the sense explained in § 2.

Let **i**, **j**, **k** denote the unit vectors along OX, OY, OZ respectively. Let **r** be the position vector of the point P whose co-ordinates are (a, b, c). Then we have

$$\vec{OP}=\vec{ON}+\vec{NP}$$
$$=\vec{OA}+\vec{AN}+\vec{NP}=\vec{OA}+\vec{OB}+\vec{OC}$$

i.e., $\mathbf{r}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.

The vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ is more conveniently written as (a, b, c). Hence we may write

$$\mathbf{r}=(a, b, c).$$

Thus (a, b, c) *are the co-ordinates of a point* P *if and only if the position vector of the point* P *is the vector* $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ *which is simply written as the vector* (a, b, c).

Remark 1. If (a, b, c) are the co-ordinates of a point P in space, then it is usually written as the point $P(a, b, c)$.

Remark 2. The co-ordinates (a, b, c) of the point P determined as above are called the **cartesian co-ordinates** of the point P, but for convenience we shall simply say that (a, b, c) are the co-ordinates of the point P.

Thus we see that the distances with proper signs of the origin from the points on the axes in which the planes through the given point P drawn parallel to the co-ordinate planes meet, are called the co-ordinates of the point P.

§ 4. Properties of co-ordinates of a point P.

(A) *The co-ordinates of a point P are respectively the distances with proper signs of the point P from the three co-ordinate planes.*

See figure of § 3. Through the point P draw planes parallel to the co-ordinate planes cutting the axes in the points A, B and C respectively. These three planes together with the co-ordinate planes form a rectangular parallelopiped. We have,

the perpendicular distance of the point P from the yz-plane

$$=LP=CM=OA=a$$

$=x$-co-ordinate of the point P,

the perpendicular distance of the point P from the zx-plane

$$=MP=CL=OB=b$$

$=y$-co-ordinate of the point P,

and the perpendicular distance of the point P from the xy-plane

$$=NP=AM=OC=c$$

$=z$-co-ordinate of the point P.

(B) *The co-ordinates of a point P are the distances from the origin O of the feet of the perpendiculars from the point P to the co-ordinates axes.*

See figure of § 3. Since the plane $PNAM$ is perpendicular to OX, PA (a line in this plane and cutting OX) is perpendicular to OX. Similarly PB and PC are perpendiculars to OY and OZ respectively. Thus

the x-co-ordinate of the point $P=OA$, A being the foot of the perpendicular from the point P on the x-axis.

Similarly the y-co-ordinate of $P=OB$, the z-co-ordinate of $P=OC$, where the points B and C are the feet of the perpendiculars from P on the y and z-axes respectively.

§ 5. **Octants.** The three co-ordinate planes namely yz-plane, zx-plane and xy-plane divide the space into eight parts called the octants, and to which octant the point P belongs is determined by the signs of the co-ordinates of the point P. The following table determines the signs in eight octants :

Octant	$OXYZ$	$OXY'Z$	$OXY'Z'$	$OXYZ'$	$OX'YZ'$	$OX'Y'Z$	$OX'YZ'$	$OX'Y'Z'$
x	+	+	+	+	−	−	−	−
y	+	−	−	+	+	−	+	−
z	+	+	−	−	+	+	−	−

Ex. 1. *What are the positions of the following points ?*

(*i*) (1, 2, 3), (*ii*) (1, −2, 3), (*iii*) (0, 0, −3),

(*iv*) (−1, −2, 0), (*v*) (2, 0, 0), (*vi*) (−1, −2, −3).

Sol. (i) (1, 2, 3) is a point in the octant $OXYZ$ and its distances from the co-ordinate planes yz, zx and xy are 1, 2 and 3 respectively.

(ii) (1, −2, 3) is a point in the octant $OXY'Z$ and its distances from the co-ordinate planes yz, zx and xy are 1, 2 and 3 respectively.

(iii) (0, 0, −3) is a point on OZ' *i.e.* on the −ve side of the z-axis situated at a distance 3 from the origin O.

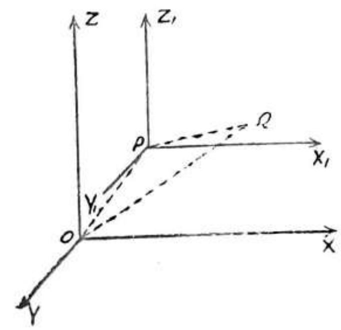
(iv) (−1, −2, 0) is a point in the co-ordinate plane xy since its z-co-ordinate is zero. It lies in the octant $OX'Y'Z$ and its distances from the co-ordinate planes yz and zx are 1 and 2 respectively.

(v) (2, 0, 0) is a point on the positive side of the x-axis situated at a distance 2 from the origin O.

(vi) (−1, −2, −3) is a point in the octant $OX'Y'Z'$ and its distances from the co-ordinate planes yz, zx and xy are 1, 2 and 3 respectively.

§ 6. **Change of origin.** Let OX, OY, OZ be a rectangular set of axes. Referred to these axes let the co-ordinates of two points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Suppose we want to shift the origin from O to the point P, *i.e.* we want to find the co-ordinates of Q referred to P as origin.

Draw the new axis PX_1, PY_1 and PZ_1 parallel to the original axes OX, OY and OZ respectively.



The position vectors of the points P and Q with respect to O as origin are given by

$$\vec{OP}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k},$$

$$\vec{OQ}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}.$$

Also the position vector of the point Q with respect to P as origin is $\vec{PQ}$. Now we have

$$\vec{PQ}=\vec{OQ}-\vec{OP}=(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})-(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})$$

$$=(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}+(z_2-z_1)\,\mathbf{k}$$

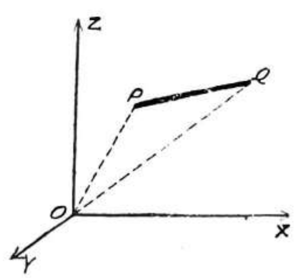
$$=(x_2-x_1,\ y_2-y_1,\ z_2-z_1).$$

Therefore the co-ordinates of the point Q with respect to the new origin P are $(x_2-x_1, y_2-y_1, z_2-z_1)$.

§ 7. The distance between two given points.

Let P and Q be two given points in space.

Let the co-ordinates of the points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) with respect to a set OX, OY, OZ of rectangular axes. The position vectors of the points P and Q are given by



$$\vec{OP}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}.$$

and $\vec{OQ}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$

Now we have $\vec{PQ}=\vec{OQ}-\vec{OP}$.

$$=(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})-(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})$$

$$=(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}+(z_2-z_1)\,\mathbf{k}.$$

$$\therefore\quad PQ=|\vec{PQ}|=\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}}.$$

Thus the distance PQ between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

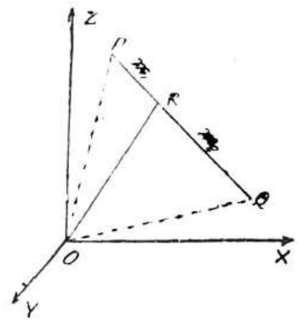
$$PQ=\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}}.$$

Corollary. The distance between the points (0, 0, 0) and (x_1, y_1, z_1) is $=\sqrt{(x_1^2+y_1^2+z_1^2)}$.

§ 8. Division of a line. *To determine the co-ordinates of a point R which divides the join of the line joining the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m_1 : m_2$.*

Let OX, OY, OZ be a set of rectangular axes.

The position vectors of the two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by



$$\vec{OP}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}\quad\ldots(1)$$

and

$$\vec{OQ}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}\quad\ldots(2)$$

Also if the co-ordinates of the point R are (x, y, z), then

$$\vec{OR}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.\quad\ldots(3)$$

Now the point R divides the join of P and Q in the ratio $m_1 : m_2$, so that

$$\frac{m_1}{m_2}=\frac{PR}{RQ}\quad\text{or}\quad m_1(RQ)=m_2(PR).$$

Hence $\quad m_2\vec{PR}=m_1\vec{RQ}$

or $\quad m_2(\vec{OR}-\vec{OP})=m_1(\vec{OQ}-\vec{OR})$

or $\quad (m_1+m_2)\,\vec{OR}=m_1\,\vec{OQ}+m_2\,\vec{OP}$

or $\quad \vec{OR}=\dfrac{m_1\,\vec{OQ}+m_2\,\vec{OP}}{m_1+m_2}$

or $\quad x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=\dfrac{(m_1x_2+m_2x_1)\,\mathbf{i}+(m_1y_2+m_2y_1)\,\mathbf{j}+(m_1z_2+m_2z_1)\,\mathbf{k}}{(m_1+m_2)}.$

[using (1), (2) and (3)]

Comparing the coefficients of **i**, **j**, **k**, we get

$$x=\frac{m_1x_2+m_2x_1}{m_1+m_2},\ y=\frac{m_1y_2+m_2y_1}{m_1+m_2},\ z=\frac{m_1z_2+m_2z_1}{m_1+m_2}.$$

Cor. 1. The middle point of the segment PQ is obtained by putting $m_1=m_2$. Hence the co-ordinates of the middle point of PQ are $(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2), \frac{1}{2}(z_1+z_2))$.

Cor. 2. If $m_1 : m_2=\mu : 1$, then the co-ordinates of the point R are $\left(\dfrac{x_1+\mu x_2}{\mu+1}, \dfrac{y_1+\mu y_2}{\mu+1}, \dfrac{z_1+\mu z_2}{\mu+1}\right)$.

These are called the **general co-ordinates of a point on the line PQ.**

Cor. 3. If the ratio (m_1/m_2) is +ve then the point R divides PQ internally and if it is −ve then externally.

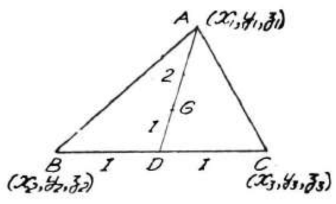
For direct applications, the co-ordinates of the point R which divides externally the join of P and Q in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1x_2-m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}, \frac{m_1z_2-m_2z_1}{m_1-m_2}\right).$$

§ 9. (A) **Centroid of a triangle.** Let ABC be a triangle, Let the co-ordinates of the vertices A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively. Let AD be a median of the $\triangle ABC$. Thus D is the mid. point of BC.

∴ The co-ordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right).$$



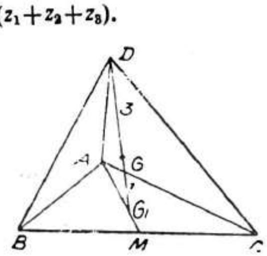
Now if G is the centroid (*i.e.*, centre of gravity) of $\triangle ABC$, then G divides AD in the ratio 2 : 1. Let the co-ordinates of G be $(\bar{x}, \bar{y}, \bar{z})$. Then

$$\bar{x}=\frac{2.\left(\frac{x_2+x_3}{2}\right)+1.x_1}{2+1}, \text{ or } \bar{x}=\frac{x_1+x_2+x_3}{3}.$$

Similarly $\bar{y}=\frac{1}{3}(y_1+y_2+y_3)$, $\bar{z}=\frac{1}{3}(z_1+z_2+z_3)$.

(B) **Centroid of a tetrahedron.**

Let $ABCD$ be a tetrahedron, the co-ordinates of whose vertices are (x_r, y_r, z_r), $r=1, 2, 3, 4$.



Let G_1 be the centroid of the face ABC of the tetrahedron. Then the co-ordinates of G_1 are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

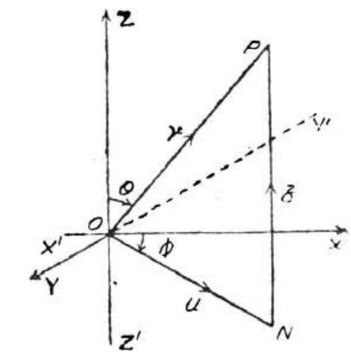
The fourth vertex D of the tetrahedron does not lie in the plane of $\triangle ABC$. We know from statics that the centroid of the tetrahedron divides the line DG_1 in the ratio 3 : 1. Let G be the centroid of the tetrahedron and if $(\bar{x}, \bar{y}, \bar{z})$ are its co-ordinates, then

$$\bar{x}=\frac{3.\frac{x_1+x_2+x_3}{3}+1.x_4}{3+1}, \text{ or } \bar{x}=\frac{x_1+x_2+x_3+x_4}{4}.$$

Similarly $\bar{y}=\frac{1}{4}(y_1+y_2+y_3+y_4)$, $\bar{z}=\frac{1}{4}(z_1+z_2+z_3+z_4)$.

§ 10. (A) **Spherical polar co-ordinates.**

Let $X'OX$, $Y'OY$ and $Z'OZ$ be the set of rectangular axes. Let P be a point in space. Draw PN perpendicular from P to the xy-plane. The position of P is determined if the length OP, angles ZOP and XON are known. Suppose $OP=r$, $\angle ZOP=\theta$ and $\angle XON=\phi$, measured positively in the directions shown by arrows in the figure. The quantities r, θ, ϕ, defined as above, are called the **spherical polar co-ordinates** of P and are written as (r, θ, ϕ).



Now we shall find relations between these co-ordinates and cartesian co-ordinates. Let (x, y, z) be the cartesian co-ordinates of P. Hence we have

$z=PN=OP\cos(\angle OPN)=r\cos(\angle ZOP)=r\cos\theta$. ...(1)

Also $ON=OP\sin\angle OPN=r\sin\theta$ [∵ $\angle ONP=90°$]

∴ $x=ON\cos\phi=r\cos\phi\sin\theta$, ...(2)

and $y=ON\sin\phi=r\sin\phi\sin\theta$. ...(3)

Thus relations (2), (3) and (1) give the relations between x, y, z and r, θ, ϕ.

Now squaring the relations (2) and (3) and adding, we get

$x^2+y^2=ON^2$, or $u^2=x^2+y^2$ where $u=ON$

or $\sqrt{(x^2+y^2)}=u=r\sin\theta$. ...(4)

Dividing (4) by (1), we get $\tan\theta=\sqrt{(x^2+y^2)}/z$.

Dividing (3) by (2), we get $\tan\phi=y/x$.

Squaring (1) and (4) and adding, we get $x^2+y^2+z^2=r^2$.

Thus the relations between spherical polar co-ordinates and cartesian co-ordinates are

$x=r\cos\phi\sin\theta$, $y=r\sin\phi\sin\theta$, $z=r\cos\theta$

$x^2+y^2+z^2=r^2$, $\tan\theta=\sqrt{(x^2+y^2)}/z$, $\tan\phi=y/x$.

(B) Cylindrical co-ordinates. See figure of § 10 (A). Let P be a point in space. The position of P can also be determined if the measure of ON, $\angle XON$ and NP are known. Suppose $ON=u$, $\angle XON=\phi$, $NP=z$. The quantities u, ϕ, z are called the cyclindrical co-ordinates of P and are written as (u, ϕ, z).

Let (x, y, z) be the cartesian co-ordinates of P, then N has the co-ordinates $(x, y, 0)$. Hence, we have

$$x=ON \cos \phi=u \cos \phi,\ y=u \sin \phi,\ z=z.$$

Also $u^2=x^2+y^2$, $\tan \phi=y/x$.

Solved Examples

Ex. 1. *P is a variable point and the co-ordinates of two points A and B are $(-2, 2, 3)$ and $(13, -3, 13)$ respectively. Find the locus of P if $3PA=2PB$.*

Sol. Let the co-ordinates of P be (x, y, z).

$\therefore\ PA=\sqrt{\{(x+2)^2+(y-2)^2+(z-3)^2\}}$, ...(1)

and $PB=\sqrt{\{(x-13)^2+(y+3)^2+(z-13)^2\}}$. ...(2)

Now it is given that $3PA=2PB$ *i.e.*, $9PA^2=4PB^2$. ...(3)

Putting the values of PA and PB from (1) and (2) in (3), we get

$$9\{(x+2)^2+(y-2)^2+(z-3)^2\}=4\{(x-13)^2+(y+3)^2+(z-13)^2\}$$

or $9\{x^2+y^2+z^2+4x-4y-6z+17\}=4\{x^2+y^2+z^2-26x+6y-26z+347\}$

or $5x^2+5y^2+5z^2+140x-60y+50z-1235=0$

or $x^2+y^2+z^2+28x-12y+10z-24=0$.

This is the required locus of P.

Ex. 2. *A, B, C are three points on the axes of x, y and z respectively at distances a, b, c from the origin O; find the co-ordinates of the point which is equidistant from A, B, C and O.*

Sol. Let the required point be $P(x, y, z)$.

The co-ordinates of the points A, B, C and O are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and $(0, 0, 0)$ respectively.

It is given that $PA=PB=PC=PO$.

Taking $PA=PO$, or $PA^2=PO^2$, we have

$(x-a)^2+y^2+z^2=x^2+y^2+z^2$

or $-2ax+a^2=0$, or $x=a/2$. ...(1)

Now taking $PB=PO$, or $PB^2=PO^2$, we have

$x^2+(y-b)^2+z^2=x^2+y^2+z^2$

or $-2by+b^2=0$, or $y=b/2$ ...(2)

Again taking $PC=PO$, or $PC^2=PO^2$, we get $z=c/2$. ...(3)

Hence the co-ordinates of the required point P are $(a/2, b/2, c/2)$.

Ex. 3. *Find the centre of the sphere which passes through the points $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.*

Sol. Let the centre of the sphere be $P(x, y, z)$.

Since the sphere passes through A, B, C and O, so $P(x, y, z)$ is equidistant from A, B, C and O. $\therefore\ PA=PB=PC=PO$.

Now proceeding as in Ex. 2 above, the centre P of the sphere is given by $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$.

Ex. 4. *Show that $(0, 7, 10)$, $(-1, 6, 6)$, $(-4, 9, 6)$ form an isosceles right angled triangle.*

Sol. Let ABC be a given triangle and let the co-ordinates of the vertices A, B and C be $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ respectively. $\therefore\ AB=\sqrt{\{(0+1)^2+(7-6)^2+(10-6)^2\}}=\sqrt{(18)}$

$BC=\sqrt{\{(-1+4)^2+(6-9)^2+(6-6)^2\}}=\sqrt{(18)}$

$CA=\sqrt{\{(-4-0)^2+(9-7)^2+(6-10)^2\}}=\sqrt{(36)}$.

We have $AB=BC$, hence $\triangle ABC$ is an isosceles triangle.

Again $AB^2+BC^2=18+18=36=CA^2$. $\therefore\ \angle ABC=90°$.

Hence $\triangle ABC$ is also right angled triangle. Therefore, the given triangle is an isosceles right angled triangle.

Ex. 5. *Find the co-ordinates of the point which divides the join of $(2, 3, 4)$ and $(3, -4, 7)$ in the ratio $2 : -4$.*

Sol. Let the co-ordinates of the required point be (x, y, z), then by § 8, we have

$$x=\frac{2(3)-4(2)}{2-4}=\frac{6-8}{-2}=1;\ y=\frac{2(-4)-4(3)}{2-4}=\frac{-20}{-2}=10;$$

$$z=\frac{2(7)-4(4)}{2-4}=\frac{14-16}{-2}=\frac{-2}{-2}=1.$$

Hence the required point is $(1, 10, 1)$.

Ex. 6. *A point P lies on the line whose end points are $A(1, 2, 3)$ and $B(2, 10, 1)$. If z co-ordinate of P is 7, find its other co-ordinates.*

Sol. Let the co-ordinates of the point P be (x, y, z) and let it divide the join of $A(1, 2, 3)$ and $B(2, 10, 1)$ in the ratio $\mu : 1$.

Then [by cor. 2, § 8] $z=\dfrac{\mu(1)+1(3)}{\mu+1}$.

But it is given that the z-co-ordinate of the point P is 7.

$\therefore\ 7=\dfrac{\mu+3}{\mu+1}$ or $7\mu+7=\mu+3$ or $6\mu=-4$ or $\mu=-2/3$.

$\therefore \quad x=\dfrac{\mu\,(2)+1\,(1)}{\mu+1}=\dfrac{(-4/3)+1}{(-2/3)+1}=-1$

and $\quad y=\dfrac{\mu\,(10)+1\,(2)}{\mu+1}=\dfrac{-(20/3)+2}{-(2/3)+1}=-14.$

Ex. 7. *Find the ratio in which the xy-plane divides the join of* $(-3, 4, -8)$ *and* $(5, -6, 4)$. *Also find the point of intersection of the line with the plane.*

Sol. Let the xy-plane (*i.e.*, $z=0$ plane) divide the line joining the points $(-3, 4, -8)$ and $(5, -6, 4)$ in the ratio $\mu : 1$, in the point R. Therefore, the co-ordinates of the point R are

[See cor. 2, § 8] $\left(\dfrac{5\mu-3}{\mu+1}, \dfrac{-6\mu+4}{\mu+1}, \dfrac{4\mu-8}{\mu+1}\right)$. ...(1)

But on xy-plane, the z co-ordinate of R is zero.

$\therefore \quad (4\mu-8)/(\mu+1)=0, \quad$ or $\quad \mu=2.$

Hence $\quad \mu : 1=2 : 1$. Thus the required ratio is $2 : 1$.

Again putting $\mu=2$ in (1), the co-ordinates of the point R become $(7/3, -8/3, 0)$.

Ex. 8. *Find the ratios in which the sphere* $x^2+y^2+z^2=504$, *divides the line joining the points* $(12, -4, 8)$ *and* $(27, -9, 18)$.

Sol. Let the sphere $x^2+y^2+z^2=504$ meet the line joining the given points in the point (x_1, y_1, z_1).

Then $\quad x_1^2+y_1^2+z_1^2=504.$ ...(1)

Now suppose that the point (x_1, y_1, z_1) divides the join of the points $(12, -4, 8)$ and $(27, -9, 18)$ in the ratio $\mu : 1$.

Then $\quad x_1=\dfrac{27\mu+12}{\mu+1},\ y_1=\dfrac{-9\mu-4}{\mu+1},\ z_1=\dfrac{18\mu+8}{\mu+1}.$

Putting the values of x_1, y_1, z_1 in (1), we get

$$\frac{(27\mu+12)^2}{(\mu+1)^2}+\frac{(-9\mu-4)^2}{(\mu+1)^2}+\frac{(18\mu+8)^2}{(\mu+1)^2}=504$$

or $\quad 9\,(9\mu+4)^2+(9\mu+4)^2+4\,(9\mu+4)^2=504\,(\mu+1)^2$

or $\quad 14\,(9\mu+4)^2=504\,(\mu+1)^2$, or $(9\mu+4)^2=36\,(\mu+1)^2$.

Taking the square root, we get $9\mu+4=\pm 6\,(\mu+1)$.

Taking +ve sign, $3\mu=2$, or $\mu/1=2/3$, or $\mu : 1=2 : 3$.

Again taking −ve sign, $15\mu=-10$, or $\mu/1=-2/3$,

or $\quad \mu : 1=2 : -3.$

Ex. 9. *From the point* $(1, -2, 3)$ *lines are drawn to meet the sphere* $x^2+y^2+z^2=4$ *and they are divided in the ratio* $2 : 3$. *Prove that the points of section lie on the sphere*

$$5\,(x^2+y^2+z^2)-6\,(x-2y+3z)+22=0.$$

Sol. Suppose any line through the given point $(1, -2, 3)$ meets the sphere $x^2+y^2+z^2=4$ in the point (x_1, y_1, z_1). Then

$$x_1^2+y_1^2+z_1^2=4. \quad ...(1)$$

Now let the co-ordinates of the point which divides the join of $(1, -2, 3)$ and (x_1, y_1, z_1) in the ratio $2 : 3$ be (x_2, y_2, z_2). Then we have

$$\left.\begin{aligned} x_2&=\frac{2.x_1+3.1}{2+3} &\text{or}\quad x_1&=\frac{5x_2-3}{2}\\ y_2&=\frac{2.y_1+3.(-2)}{2+3} &\text{or}\quad y_1&=\frac{5y_2+6}{2}\\ z_2&=\frac{2.z_1+3.3}{2+3} &\text{or}\quad z_1&=\frac{5z_2-9}{2}\end{aligned}\right\} \quad ...(2)$$

Putting the values of x_1, y_1, z_1, from (2) in (1), we have

$$(5x_2-3)^2+(5y_2+6)^2+(5z_2-9)^2=4\times 4$$

or $\quad 25\,(x_2^2+y_2^2+z_2^2)-30x_2+60y_2-90z_2+110=0$

or $\quad 5\,(x_2^2+y_2^2+z_2^2)-6\,(x_2-2y_2+3z_2)+22=0.$

$\therefore \quad$ the locus of (x_2, y_2, z_2) is

$$5\,(x^2+y^2+z^2)-6\,(x-2y+3z)+22=0,$$

which is the equation of a sphere.

Ex. 10. *Prove that the three points A, B and C whose co-ordinates are* $(3, -2, 4), (1, 1, 1)$ *and* $(-1, 4, -2)$ *respectively, are collinear.*

Solution. The general co-ordinates of a point R which divides the line joining $A\,(3, -2, 4)$ and $B\,(1, 1, 1)$ in the ratio $\mu : 1$ are

$$\left(\frac{\mu+3}{\mu+1}, \frac{\mu-2}{\mu+1}, \frac{\mu+4}{\mu+1}\right). \quad ...(1)$$

If $C\,(-1, 4, -2)$ also lies on the line AB, then for some value of μ the co-ordinates of the point R will be the same as those of C.

$\therefore$ let the x-co-ordinate of the point R = the x-co-ordinate of the point C.

Then $(\mu+3)/(\mu+1)=-1, \quad$ or $\quad \mu=-2.$

Putting $\mu=-2$ in (1), the co-ordinates of R are $(-1, 4, -2)$ which are also the co-ordinates of C. Hence the points A, B and C are collinear.

Also we note that C divides AB in the ratio $\mu : 1$, *i.e.*, $-2 : 1$.

Exercises

1. Find the locus of a point P which moves in such a way that its distance from the point $A\,(u, v, w)$ is always equal to a.

   Ans. $x^2+y^2+z^2-2ux-2vy-2wz+u^2+v^2+w^2-a^2=0$.

2. The axes are rectangular and A, B are the points $(3, 4, 5)$, $(-1, 3, -7)$. A variable point P moves such that (i) $PA=PB$ and (ii) $PA^2-PB^2=2k^2$. Find the locus of P in each of the above cases.

   Ans. (i) $8x+2y+24z+9=0$. (ii) $8x+2y+24z+9+2k^2=0$.

3. Show that the points $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$ form an equilateral triangle.

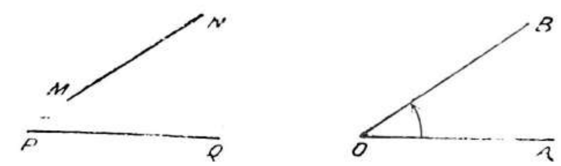
   Hint. Show that the length of each side of the triangle is $\sqrt{6}$.

4. Prove that the three points A, B and C whose coordinates are $(3, 2, -4)$, $(5, 4, -6)$ and $(9, 8, -10)$ respectively are collinear.

2
Direction Cosines and Projections

§ 1. Angle between two non-coplanar (i.e. non-intersecting lines).

Let PQ and MN be two non-coplanar lines. The angle between two non-coplanar lines PQ and MN is equal to the angle between two straight lines OA and OB drawn from any point O parallel to PQ and MN respectively. Thus the angle between

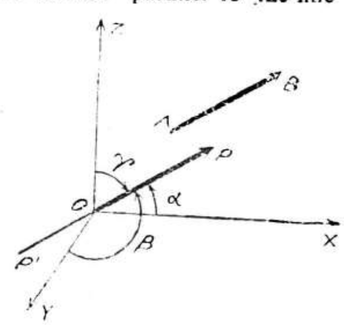


the lines PQ and MN is equal to the angle AOB.

§ 2. Direction cosines of a line.

Definition. *If α, β, γ are the angles which a given directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines (briefly written as d.c.'s) of the line. These d.c.'s are usually denoted by l, m, n.*

Let AB be a given line. Draw a line OP parallel to the line AB and passing through the origin O. Measure angles α, β, γ as shown by arrows in the figure, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the d.c.'s of the line AB. It can be easily seen *that l, m, n are the direction cosines of a line if and only if $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$ is a unit vector in the direction of that line.*



Clearly OP' (*i.e.*, the line through O and parallel to BA) makes angles $180°-\alpha$, $180°-\beta$,

$180°-\gamma$ with OX, OY and OZ respectively. Hence d.c.'s of the line BA are $\cos(180°-\alpha)$, $\cos(180°-\beta)$, $\cos(180°-\gamma)$ *i.e.*, are $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$.

Remark. Since the angles α, β, γ are not coplanar,

$$\therefore \quad \alpha+\beta+\gamma \neq 360°.$$

D.c.'s of the coordinate axes.

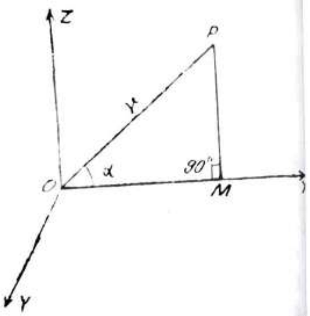
Since the axis of x makes angles $0°, 90°, 90°$ with the axes of x, y, z respectively, therefore by definition, its d.c.'s are $\cos 0°$, $\cos 90°$, $\cos 90°$ *i.e.*, 1, 0, 0.

Hence the **d.c.'s of the x-axis are 1, 0, 0.**

Similarly the **d.c.'s of the y-axis are 0, 1, 0**

and the **d.c.'s of the z-axis are 0, 0, 1.**

§ 3. *If the length of a line OP through the origin O be r, then the co-ordinates of P are (lr, mr, nr) where l, m, n, are the d.c.'s of OP.*



Draw PM perpendicular from P to OX meeting it at M. Let (x, y, z) be the co-ordinates of P, then $OM=x$. From the right angled $\triangle OMP$, we have $\dfrac{OM}{OP}=\cos\alpha=l$.

or $\quad x/r=l$ or $x=lr$.

Similarly $y=mr$, $z=nr$.

$\therefore$ P is the point (lr, mr, nr).

§ 4. *If l, m, n are direction cosines of any line AB, then to prove that $l^2+m^2+n^2=1$.* **(Kanpur 1983)**

Through the origin O draw a line OP parallel to the given line AB so that the d.c.'s of OP are l, m, n. Suppose OP is of length r. If the co-ordinates of P are (x, y, z), then we have (see § 3)
$$x=lr, \; y=mr, \; z=nr. \qquad \ldots(1)$$

Now $r^2=OP^2$ or $r^2=(x-0)^2+(y-0)^2+(z-0)^2$

or $\quad r^2 \Rightarrow x^2+y^2+z^2$ or $r^2=l^2r^2+m^2r^2+n^2r^2$, $\quad$ using (1),

or $\quad l^2+m^2+n^2=1.$

Remark. We have $\vec{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=lr\mathbf{i}+mr\mathbf{j}+nr\mathbf{k}$.

$\therefore$ a unit vector in the direction of OP

$$=\frac{\vec{OP}}{r}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k} \qquad [\because \; |\vec{OP}|=r] \qquad \ldots(2)$$

The relation (2) shows that the direction cosines of the line OP are the coefficients of **i, j, k** *in the rectangular resolution of the unit vector in the direction of OP.*

Thus if l, m, n are the d.c.'s of a line, then a unit vector along that line is $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$.

$$\therefore \quad |l\mathbf{i}+m\mathbf{j}+n\mathbf{k}|=1$$

or $\quad \sqrt{(l^2+m^2+n^2)}=1$, or $l^2+m^2+n^2=1$.

§ 5. **Direction ratios.**

Definition. *If the direction cosines l, m, n of a given line be proportional to any three numbers a, b, c respectively, then the numbers a, b, c are called direction ratios (briefly written as d.r.'s) of the given line.*

Relation between direction cosines and direction ratios.

Let a, b, c be the direction ratios of a line whose d.c.'s are l, m, n. From the definition of d.r.'s, we have

$l/a=m/b=n/c=k$ (say). Then $l=ka$, $m=kb$, $n=kc$.

But $\quad l^2+m^2+n^2=1.$

$\therefore \quad k^2(a^2+b^2+c^2)=1$, or $k^2=1/(a^2+b^2+c^2)$

or $\quad k=\pm 1/\sqrt{(a^2+b^2+c^2)}$.

Taking the positive value of k, we get

$$l=\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \; m=\frac{b}{\sqrt{(a^2+b^2+c^2)}}, \; n=\frac{c}{\sqrt{(a^2+b^2+c^2)}}.$$

Again taking the negative value of k, we get

$$l=\frac{-a}{\sqrt{(a^2+b^2+c^2)}}, \; m=\frac{-b}{\sqrt{(a^2+b^2+c^2)}}, \; n=\frac{-c}{\sqrt{(a^2+b^2+c^2)}}.$$

Remark. Direction cosines of a line are unique. But the direction ratios of a line are by no means unique. If a, b, c are direction ratios of a line, then ka, kb, kc are also direction ratios of that line where k is any non-zero real number. Moreover if a, b, c are direction ratios of a line, then $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ is a vector parallel to that line.

Rule. *Let a, b, c be the d.r.'s of a given line, then, to find actual direction cosines of this line, divide each of a, b, c by*

$$\sqrt{(a^2+b^2+c^2)}.$$

SOLVED EXAMPLES 2 (A)

Ex. 1. *Find the d.c.'s of a line whose direction ratios are* 2, 3, -6.

Solution. We have $\sqrt{\{(2)^2+(3)^2+(-6)^2\}}=\sqrt{(4+9+36)}=7$.

Hence (by § 5) the d.c.'s of the given line are $\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{-6}{7}$.

Ex. 2. *Prove that* $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$, *where* α, β, γ *are the angles which the given line makes with positive directions of the axes.* **(Agra 1978)**

Sol. We have (See § 4) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

Changing cosines into sines, we get

$$(1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma)=1$$

or $$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2.$$

Ex. 3. *Find the direction cosines of the line which is equally inclined to the axes.* **(Agra 1979)**

Sol. Let the direction cosines of the line be l, m, n. Since the line is equally inclined to the co-ordinate axes, therefore

$$l^2=m^2=n^2.$$

But $l^2+m^2+n^2=1$. $\therefore$ $3l^2=1$ or $l^2=1/3$ or $l=\pm 1/\sqrt{3}$.

Similarly $m=\pm 1/\sqrt{3}$, $n=\pm 1/\sqrt{3}$.

Hence the direction cosines of the line are $\pm 1/\sqrt{3}$, $\pm 1/\sqrt{3}$, $\pm 1/\sqrt{3}$. There will be eight such lines, one lying in each octant.

Ex. 4. *Find the direction cosines l, m, n of two lines which are connected by the relations* $l-5m+3n=0$ *and* $7l^2+5m^2-3n^2=0$.

Sol. The given relations are

$$l-5m+3n=0 \text{ or } l=5m-3n \quad ...(1)$$

and $$7l^2+5m^2-3n^2=0. \quad ...(2)$$

Putting the value of l from (1) in (2), we get

$$7(5m-3n)^2+5m^2-3n^2=0$$

or $$180m^2-210mn+60n^2=0 \text{ or } 6m^2-7mn+2n^2=0$$

or $$6m^2-4mn-3mn+2n^2=0 \text{ or } (2m-n)(3m-2n)=0.$$

$\therefore$ $m/n=\frac{1}{2}$, and $m/n=2/3$.

Now when $m/n=1/2$ *i.e.* $n=2m$, we have from the relation (1)

$$l=5m-6m \text{ or } l=-m \text{ or } \frac{l}{m}=-\frac{1}{1}.$$

Thus $\frac{m}{n}=\frac{1}{2}$ and $\frac{l}{m}=-\frac{1}{1}$ giving $\frac{l}{-1}=\frac{m}{1}=\frac{n}{2}$

or $$\frac{l}{-1}=\frac{m}{1}=\frac{n}{2}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(-1)^2+1^2+2^2\}}}=\frac{1}{\sqrt{6}}.$$

$\therefore$ The d.c.'s of one line are $-1/\sqrt{6}$, $1/\sqrt{6}$, $2/\sqrt{6}$.

Again when $\frac{m}{n}=\frac{2}{3}$ *i.e.* $n=\frac{3m}{2}$, we have from (1)

$$l=5m-\frac{9m}{2} \text{ or } l=\frac{m}{2} \text{ or } \frac{l}{1}=\frac{m}{2}.$$

Thus $\frac{m}{n}=\frac{2}{3}$ and $\frac{l}{1}=\frac{m}{2}$ giving

$$\frac{l}{1}=\frac{m}{2}=\frac{n}{3}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(1^2+2^2+3^2)}}=\frac{1}{\sqrt{(14)}}.$$

$\therefore$ The d.c.'s of the other line are $1/\sqrt{(14)}$, $2/\sqrt{(14)}$, $3/\sqrt{(14)}$.

Ex. 5. *Find the direction cosines l, m, n of the two lines which are connected by the relations* $l+m+n=0$ *and* $mn-2nl-2lm=0$. **(Meerut 1985)**

Sol. The given relations are

$$l+m+n=0 \text{ or } l=-m-n \quad ...(1)$$

and $$mn-2nl-2lm=0. \quad ...(2)$$

Putting the value of l from (1) in the relation (2), we get

$$mn-2n(-m-n)-2(-m-n)m=0 \text{ or } 2m^2+5mn+2n^2=0$$

or $$(2m+n)(m+2n)=0.$$

$$\therefore \quad \frac{m}{n}=-\frac{1}{2} \text{ and } -2.$$

From (1), we have $\frac{l}{n}=\frac{-m-n}{n}=-\frac{m}{n}-1$. ...(3)

Now when $m/n=-\frac{1}{2}$, (3) gives $l/n=\frac{1}{2}-1=-\frac{1}{2}$.

$$\therefore \quad m/1=n/-2 \text{ and } l/1=n/-2$$

i.e., $$\frac{l}{1}=\frac{m}{1}=\frac{n}{-2}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{1^2+1^2+(-2)^2\}}}=\frac{1}{\sqrt{6}}.$$

$\therefore$ The d.c.'s of one line are $1/\sqrt{6}$, $1/\sqrt{6}$, $-2/\sqrt{6}$.

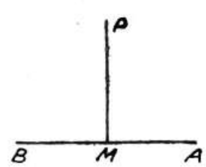
Again when $m/2=-2$, (3) gives $l/n=2-1=1$.

$$\therefore \quad \frac{l}{1}=\frac{m}{-2}=\frac{n}{1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{1^2+(-2)^2+1^2\}}}=\frac{1}{\sqrt{6}}.$$

$\therefore$ The d.c.'s of the other line are $1/\sqrt{6}$, $-2/\sqrt{6}$, $1/\sqrt{6}$.

§ 6. Projection of a point on a given line.

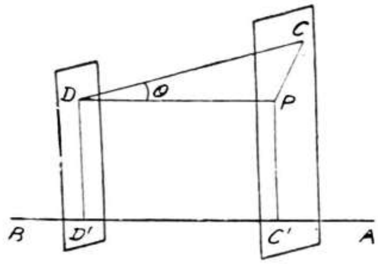
Let P be a given point and AB the given straight line. Draw perpendicular PM from P to AB, meeting AB in M. Then the foot of the perpendicular M is called the projection of the given point P on the given line AB.



The point M is the point in which the plane through P and perpendicular to AB meets the line AB.

§ 7. Projection of a segment of a line on another line (in the same plane or another).

Suppose we are to find the projection of a segment CD of a line on another given line AB. Let the points C', D' be the projections of the points C, D on the line AB, then the segment



$C'D'$ is the required projection of the segment CD. The projection of the segment CD on the line AB may also be defined as the intercept $C'D'$ made on the line AB by the planes through the points C and D each perpendicular to AB.

To find the length of the projection C'D'. Through D draw a line DP parallel to $D'C'$ and meeting the plane through C perpendicular to AB in P. Thus $DP = D'C'$. ...(1)

Let θ be the angle between the lines AB and CD. Also AB is parallel to PD and hence $\angle CDP = \theta$. Again the line CP lies in the plane which is perpendicular to AB and hence CP is perpendicular to DP. $\therefore$ $\angle DPC = 90°$.

Therefore $DP = DC \cos\theta$. ...(2)

From (1) and (2), we have $D'C' = DC \cos\theta$

or $C'D' = CD \cos\theta$.

Remark. Let $\overrightarrow{DC} = \mathbf{a}$ and let $\mathbf{b}$ be a unit vector along BA. Then

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, by definition of dot product of two vectors

$= DC\cos\theta$ $[\because |\mathbf{b}| = 1 \text{ and } |\mathbf{a}| = DC]$

= projection of DC on BA.

Thus to get the projection of DC on a line BA, we take the dot product of the vector $\overrightarrow{DC}$ with a unit vector along BA.

§ 8. Projection of a broken line an a given line, Or Given n points $C_1, C_2, \ldots, C_n$ (say) in space, to find the projection of C_1C_n on a given line.

Suppose AB is a given line. Let the projections of the n points $C_1, C_2, \ldots, C_n$ on the given line AB be the points

$M_1, M_2, \ldots, M_n$ respectively. Thus the points $M_1, M_2, \ldots, M_n$ lie on the same straight line AB. Hence in view of § 7, we have

projection of $C_1C_2 = M_1M_2$,

projection of $C_2C_3 = M_2M_3, \ldots$,

and projection of $C_{n-1}C_n = M_{n-1}M_n$.

$\therefore$ sum of the projections of $C_1C_2, C_2C_3, \ldots, C_{n-1}C_n$ on the line $AB = M_1M_2 + M_2M_3 + \ldots + M_{n-1}M_n$

$= M_1M_n =$ projection of C_1C_n on AB.

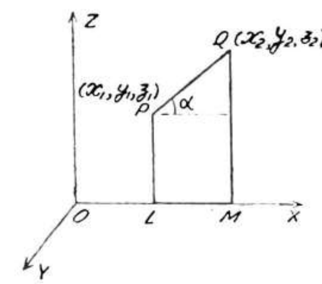
Hence we conclude that :

Projection of C_1C_n on a line AB = Sum of the projections of $C_1C_2, C_2C_3, \ldots, C_{n-1}C_n$ on the line AB.

§ 9. Direction cosines of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2).

Let the line PQ make angles α, β, γ with x, y, z axes respectively. If l, m, n be the d.c.'s of this line, then

$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.



Now if L and M are the projections of the points P and Q respectively on the x-axis, then we have

$OL = x_1$, $OM = x_2$, so that

$LM = OM - OL = x_2 - x_1$.

But LM is the projection of PQ on the axis of x. Hence by § 7, we have $LM = PQ\cos\alpha$ or $x_2 - x_1 = l.PQ$

or $(x_2 - x_1)/l = PQ$. ...(1)

In a similar way by projecting PQ on y and z axes, we have

$(y_2 - y_1)/m = PQ$ and $(z_2 - z_1)/n = PQ$. ...(2)

So from (1) and (2), we have

$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ.$$

Hence we conclude that the **actual direction cosines** *l, m, n of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are*

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

respectively, where $PQ = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$.

Again the direction cosines of the line PQ are proportional to

$x_2-x_1, y_2-y_1, z_2-z_1$ *which are therefore the* **direction ratios** *of the line PQ.*

Vector method. Since the co-ordinates of the points P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, therefore

$\vec{OP}=$ the position vector of $P=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$

and $\vec{OQ}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$.

$\therefore \quad \vec{PQ}=\vec{OQ}-\vec{OP}=(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}+(z_2-z_1)\,\mathbf{k}$.

Also $\quad |\vec{PQ}|=PQ=\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}}$.

$\therefore$ a unit vector along $PQ=\dfrac{\vec{PQ}}{PQ}=\dfrac{x_2-x_1}{PQ}\,\mathbf{i}+\dfrac{y_2-y_1}{PQ}\,\mathbf{j}+\dfrac{z_2-z_1}{PQ}\,\mathbf{k}$.

Hence the d.c.'s of the line PQ are

$$\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}.$$

Also the d.r.'s of the line PQ are $x_2-x_1, y_2-y_1, z_2-z_1$.

§ 10. If O and P are two points (0, 0, 0) and (x_1, y_1, z_1), then to prove that the projection of OP on a line whose direction cosines are l, m, n is $lx_1+my_1+nz_1$.

Construct a rectangular parallelopiped with diagonal as OP and faces parallel to the co-ordinate planes. [See figure of § 3, chapter 1, page 3]. Then clearly we have

$$OA=x_1,\ OB=y_1=AN,\ CO=z_1=NP.$$

Now considering OP as a broken line consisting of the parts OA, AN and NP, we have

the projection of OP on a line with d.c.'s l, m, n
$=$ sum of the projections of OA, AN and NP on the line with d.c.'s l, m, n

$=lx_1+my_1+nz_1$. [by § 7]

Vector method. We have $\vec{OP}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$.

Also a unit vector along the line whose d.c.'s are l, m, n
$=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$.

$\therefore$ projection of OP on the line whose d.c.'s are l, m, n
$=(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=lx_1+my_1+nz_1$.

§ 11. To find the projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose d.c.'s are l, m, n.

Let O be the origin. Then

$\vec{OP}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$ and $\vec{OQ}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$.

$\therefore \quad \vec{PQ}=\vec{OQ}-\vec{OP}=(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}+(z_2-z_1)\,\mathbf{k}$.

Now the unit vector along the line whose d.c.'s are l, m, n
$=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$.

$\therefore$ projection of PQ on the line whose d.c.'s are l, m, n
$=[(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}+(z_2-z_1)\,\mathbf{k}]\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})$
$=l\,(x_2-x_1)+m\,(y_2-y_1)+n\,(z_2-z_1)$.

Remark. In the articles 10 and 11, l, m, n are the actual direction cosines and not direction ratios.

SOLVED EXAMPLES 2 (B)

Ex. 1. *If P (2, 3, −6) and Q (3, −4, 5) are two points, find the d.c.'s of OP, PO, OQ and PQ where O is the origin.*

Sol. By § 9, the direction ratios of OP are $2-0, 3-0, -6-0$, *i.e.*, are $2, 3, -6$.

Also $OP=\sqrt{\{2^2+3^2+(-6)^2\}}=\sqrt{(49)}=7$.

Hence d.c.'s of OP are

$$\frac{2}{OP}, \frac{3}{OP}, \frac{-6}{OP} \text{ i.e., are } \frac{2}{7}, \frac{3}{7}, \frac{-6}{7}. \qquad \textbf{Ans.}$$

The d.r.'s of PO are $0-2, 0-3, 0-(-6)$ *i.e.*, $-2, -3, 6$.

Also $PO=\sqrt{\{(-2)^2+(-3)^2+6^2\}}=7$.

$\therefore$ d.c.'s of PO are $-2/7, -3/7, 6/7$. **Ans.**

The d.r.'s of OQ are $3-0, -4-0, 5-0$ *i.e.*, $3, -4, 5$.

Also $OQ=\sqrt{\{(3)^2+(-4)^2+(5)^2\}}=\sqrt{(50)}=5\sqrt{2}$.

$\therefore$ d.c.'s of OQ are $3/(5\sqrt{2}), -4/(5\sqrt{2}), 5/(5\sqrt{2})$. **Ans.**

The d.r.'s of PQ are $3-2, -4-3, 5-(-6)$ *i.e.*, $1, -7, 11$.

Also $PQ=\sqrt{\{(3-2)^2+(-4-3)^2+(5+6)^2\}}=\sqrt{(171)}$.

$\therefore$ d.c.'s of PQ are $1/\sqrt{(171)}, -7/\sqrt{(171)}, 11/\sqrt{(171)}$. **Ans.**

Ex. 2. *Find the length of a segment of a line whose projections on the axes are* 2, 3, 6.

Sol. Let CD be a segment of a line whose direction cosines are l, m, n. Then, we have

$2=$ the projection of CD on the x-axis $=l.CD$ ...(1)
$3=$ the projection of CD on the y-axis $=m.CD$ ...(2)
$6=$ the projection of CD on the z-axis $=n.CD$. ...(3)

Squaring (1), (2), (3) and adding, we have

$$4+9+36=(l^2+m^2+n^2)\,CD^2$$

or $\quad 49=1.CD^2 \qquad [\because\ l^2+m^2+n^2=1]$

or $\quad CD=7$. **Ans.**

Ex. 3. *If P, Q, R, S are four points with co-ordinates (3, 4, 5),* (4, 6, 3), (−1, 2, 4), (1, 0, 5) *respectively, then find the projection of PQ on RS. Also find the projection of RS on PQ.*

Sol. To find the projection of PQ on RS, we should find the d.c.'s of RS.

The direction ratios of RS are

$$1-(-1),\ 0-2,\ 5-4 \text{ i.e., } 2, -2, 1.$$

Also $RS=\sqrt{[\{1-(-1)\}^2+(0-2)^2+(5-4)^2]}=\sqrt{(4+4+1)}=3.$

$\therefore$ d.c.'s of RS are $2/3, -2/3, 1/3$.

Hence the projection of PQ on RS is (See § 11)

$$=l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$$

$$=\tfrac{2}{3}(4-3)-\tfrac{2}{3}(6-4)+\tfrac{1}{3}(3-5)=\tfrac{2}{3}-\tfrac{4}{3}-\tfrac{2}{3}=-\tfrac{4}{3}. \qquad \textbf{Ans.}$$

Again to find the projection of RS on PQ, we should find the d.c.'s of PQ

The direction ratios of PQ are

$$4-3,\ 6-4,\ 3-5 \text{ i.e., } 1, 2, -2.$$

Also $PQ=\sqrt{\{(4-3)^2+(6-4)^2+(3-5)^2\}}=\sqrt{(1+4+4)}=3.$

$\therefore$ the d.c.'s of PQ are $1/3, 2/3, -2/3$.

$\therefore$ projection of RS on PQ is [See § 11]

$$=\tfrac{1}{3}\{1-(-1)\}+\tfrac{2}{3}(0-2)-\tfrac{2}{3}(5-4)=\tfrac{2}{3}-\tfrac{4}{3}-\tfrac{2}{3}=-\tfrac{4}{3}. \qquad \textbf{Ans.}$$

Ex. 4. *If P, Q, R, S are four points with co-ordinates* (2, 3, −1), (3, 5, −3), (1, 2, 3), (3, 5, 7) *respectively, prove by projections that PQ is at right angles to RS.*

Sol. In order that PQ is at right angles to RS, the projection of PQ on RS should be zero.

The direction ratios of RS are $3-1, 5-2, 7-3$, *i.e.*, 2, 3, 4.

Also $RS=\sqrt{\{(3-1)^2+(5-2)^2+(7-3)^2\}}$

$=\sqrt{(4+9+16)}=\sqrt{(29)}$.

$\therefore$ d.c.'s of RS are $2/\sqrt{(29)}, 3/\sqrt{(29)}, 4/\sqrt{(29)}$.

Hence the projection of PQ on RS is [See § 11]

$$=\frac{2}{\sqrt{(29)}}\cdot(3-2)+\frac{3}{\sqrt{(29)}}\cdot(5-3)+\frac{4}{\sqrt{(29)}}\{-3-(-1)\}$$

$$=\frac{2+6-8}{\sqrt{(29)}}=0.$$

Therefore PQ is at right angles to RS.

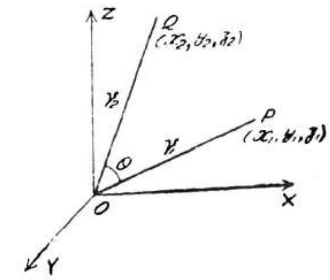
§ 12. Angle between two lines.

To show that the angle θ between any two lines whose direction cosines are l_1, m_1, n_1 *and* l_2, m_2, n_2 *is given by*

$$\cos\theta=\mathbf{l_1l_2+m_1m_2+n_1n_2}.$$

(Meerut 1980, 85S: Kanpur 82)

Let AB and CD be two given lines whose d.c.'s are l_1, m_1, n_1; and l_2, m_2, n_2 respectively. Through the origin O draw lines OP and OQ parallel to AB and CD respectively so that direction cosines of OP and OQ are l_1, m_1, n_1; and l_2, m_2, n_2 respectively. Take a point $P(x_1, y_1, z_1)$ on OP such that $OP=r_1$. Since the d.c.'s of OP are l_1, m_1, n_1, therefore the co-ordinates of P may be written as



$$(l_1r_1, m_1r_1, n_1r_1).$$

$$\therefore \quad x_1=l_1r_1,\ y_1=m_1r_1,\ z_1=n_1r_1. \qquad \ldots(1)$$

Similarly if we take a point $Q(x_2, y_2, z_2)$ on OQ such that $OQ=r_2$, then

$$x_2=l_2r_2,\ y_2=m_2r_2,\ z_2=n_2r_2. \qquad \ldots(2)$$

Let θ be the angle between the lines AB and CD, then θ is the angle between OP and OQ.

Now the projection of OQ on OP

$$=l_1(x_2-0)+m_1(y_2-0)+n_1(z_2-0)$$

$$=l_1x_2+m_1y_2+n_1z_2 \qquad \ldots(3)$$

But the projection of OQ on $OP=OQ\cos\theta=r_2\cos\theta$. ...(4)

From (3) and (4), we have $r_2\cos\theta=l_1x_2+m_1y_2+n_1z_2$

or $\quad r_2\cos\theta=l_1l_2r_2+m_1m_2r_2+n_1n_2r_2 \qquad$ using (2)

or $\quad \cos\theta=\mathbf{l_1l_2+m_1m_2+n_1n_2}. \qquad \ldots(5)$

Remark If θ is the acute angle between the two lines, then $\cos\theta$ is +ive and so we have

$$\cos\theta=|l_1l_2+m_1m_2+n_1n_2|.$$

Corresponding formula when direction ratios of the lines are given. Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of the two given lines. Then their actual direction cosines are given by

$$\frac{a_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}},\ \frac{b_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}},\ \frac{c_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}} \text{ and}$$

$$\frac{a_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}},\ \frac{b_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}},\ \frac{c_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}.$$

Using these values of d.c.'s in (5), the angle θ between these two lines is given by

$$\cos\theta=\frac{\mathbf{a_1a_2+b_1b_2+c_1c_2}}{\sqrt{(\mathbf{a_1^2+b_1^2+c_1^2})}\sqrt{(\mathbf{a_2^2+b_2^2+c_2^2})}}. \qquad \ldots(6)$$

Cor. 1. *To find sin θ and tan θ in terms of d.c.'s and d.r.'s of the two given lines.*

First of all we state the **Lagrange's identity** which is as follows :

If l_1, m_1, n_1 and l_2, m_2, n_2 are two sets of real numbers, then

$$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2$$
$$=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2. \quad ...(7)$$

[Remember]

Now, we have

$$\sin^2\theta=1-\cos^2\theta=1-(l_1l_2+m_1m_2+n_1n_2)^2$$
$$=(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2$$
$$=(m_1n_2-m_2n_1)^2+(n_1l_2+n_2l_1)^2+(l_1m_2-l_2m_1)^2, \text{ using (7).}$$
$$\therefore \quad \sin\theta=\sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}}. \quad ...(8)$$

The value of $\sin^2\theta$ may be conveniently remembered in terms of determinants as follows :

We write the corresponding direction cosines of the two lines above and below in two rows as follows ;

$$\left.\begin{matrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{matrix}\right\} \quad ..(A)$$

Then $\sin^2\theta=\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2+\begin{vmatrix} l_1 & n_1 \\ l_2 & n_2 \end{vmatrix}^2+\begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2.$

To get the first determinant we suppress the first column in (A), to get the second determinant we suppress the second column and to get the third determinant we suppress the third column.

In terms of direction ratios, the formula (8) is given by

$$\sin^2\theta=\frac{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}. \quad ...(9)$$

Again, $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{\sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}}}{l_1l_2+m_1m_2+n_1n_2}$ ...(10)

in terms of direction cosines, while in terms of direction ratios, we have

$$\tan\theta=\frac{\sqrt{\{\Sigma(b_1c_2-b_2c_1)^2\}}}{a_1a_2+b_1b_2+c_1c_2}. \quad ...(11)$$

Note that the formula for tan θ is the same whether we are given direction cosines or direction ratios.

Cor. 2. Condition for perpendicularity. If the lines are perpendicular to each other, then $\theta=90°$ *i.e.* $\cos\theta=\cos 90°=0$ and therefore from (5), the required condition is

$$\mathbf{l_1l_2+m_1m_2+n_1n_2=0.} \quad ...(12)$$

In terms of direction ratios the required condition of perpendicularity [from (6)] is

$$\mathbf{a_1a_2+b_1b_2+c_1c_2=0.} \quad ...(13)$$

Note that the condition for perpendicularity is the same whether we have direction cosines or direction ratios.

Cor. 3. Condition for parallelism : If the lines are parallel to each other then $\theta=0$ *i.e.* $\sin\theta=0$ and from (8), we have

$$(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2=0.$$

The L.H.S. being sum of three squares of real quantities will be zero if each of them is separately zero and hence we have

$$m_1n_2-m_2n_1=0,\ n_1l_2-n_2l_1=0,\ l_1m_2-l_2m_1=0$$

or $$\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}=\frac{\sqrt{(l_1^2+m_1^2+n_1^2)}}{\sqrt{(l_2^2+m_2^2+n_2^2)}}=\frac{\sqrt{(1)}}{\sqrt{(1)}}=1.$$

$$\therefore \quad \mathbf{l_1=l_2,\ m_1=m_2,\ n_1=n_2.} \quad ...(14)$$

This shows that the two lines will be parallel if their direction cosines are the same.

In terms of direction ratios the required condition of parallelism is $\dfrac{\mathbf{a_1}}{\mathbf{a_2}}=\dfrac{\mathbf{b_1}}{\mathbf{b_2}}=\dfrac{\mathbf{c_1}}{\mathbf{c_2}}.$

This shows that the two lines are parallel if their direction ratios are proportional.

Vector approach for § 12. (i) *To find the angle between two lines whose d c.'s are l_1, m_1, n_1 and l_2, m_2, n_2.*

We have **a** = the unit vector along the line whose d.c.'s are $l_1, m_1, n_1=l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k}$,

and **b** = the unit vector along the line whose d.c.'s are $l_2, m_2, n_2=l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k}$.

If θ be the angle between the two lines, then

$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\ |\mathbf{b}|\cos\theta$, by definition of dot product of two vectors

$\Rightarrow (l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k})\cdot(l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k})$
$=(1)(1)\cos\theta \quad [\because |\mathbf{a}|=1=|\mathbf{b}|]$

$\Rightarrow \cos\theta=l_1l_2+m_1m_2+n_1n_2.$

Also by definition of cross product of two vectors, we have

$\mathbf{a}\times\mathbf{b}=|\mathbf{a}|\ |\mathbf{b}|\sin\theta\,\mathbf{N}$, where **N** is a unit vector perpendicular to both **a** and **b**

$=\sin\theta\,\mathbf{N}$, because $|\mathbf{a}|=1=|\mathbf{b}|$.

$(\mathbf{a}\times\mathbf{b})^2=\sin^2\theta. \quad [\because \mathbf{N}^2=\mathbf{N}\cdot\mathbf{N}=1]$

But $\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

$$=(m_1n_2-m_2n_1)\,\mathbf{i}+(n_1l_2-n_2l_1)\,\mathbf{j}+(l_1m_2-l_2m_1)\,\mathbf{k}.$$

$\therefore\quad \sin^2\theta=(\mathbf{a}\times\mathbf{b})^2=(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})$

$$=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2.$$

Condition for perpendicularity. The two lines are perpendicular if and only if $\mathbf{a}\cdot\mathbf{b}=0$

i.e., iff $(l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k})\cdot(l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k})=0$

i.e., iff $l_1l_2+m_1m_2+n_1n_2=0$.

Condition for parallelism. The two lines are parallel if and only if the vectors **a** and **b** are collinear

i.e., iff $\mathbf{a}=\lambda\mathbf{b}$, where λ is some scalar *i.e.*, real number

i.e., iff $l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k}=\lambda\,(l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k})$

i.e., iff $l_1=\lambda l_2,\ m_1=\lambda m_2,\ n_1=\lambda n_2$

i.e., iff $\dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}$.

(ii) *To find the angle between two lines whose d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2.*

We have

$\mathbf{A}=$ a vector along the line whose d.r.'s are a_1, b_1, c_1

$=a_1\mathbf{i}+b_1\mathbf{j}+c_1\mathbf{k}$,

$\mathbf{B}=$ a vector along the line whose d.r.'s are a_2, b_2, c_2

$=a_2\mathbf{i}+b_2\mathbf{j}+c_2\mathbf{k}$.

Now proceed as in case (i). Here $|\mathbf{A}|=\sqrt{(a_1^2+b_1^2+c_1^2)}$ and $|\mathbf{B}|=\sqrt{(a_2^2+b_2^2+c_2^2)}$.

§ 3. To find the perpendicular distance of a point P (x′, y′, z′) from a line through A (a, b, c) and whose direction cosines are l, m, n

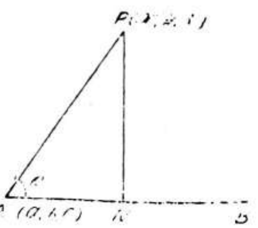
Let AB be a line through $A\,(a, b, c)$ and whose d.c.'s are l, m, n. Let PN be the perpendicular from P to AB.

Now $AN=$ projection of the line segment joining $A\,(a, b, c)$ and $P\,(x', y', z')$ on the line AB

$=(x'-a)\,l+(y'-b)\,m+(z'-c)\,n$,

and $AP=$ distance between the points A and P

$=\sqrt{[(x'-a)^2+(y'-b)^2+(z'-c)^2]}$.



We have, $PN^2=AP^2-AN^2$

or $\quad PN^2=\{(x'-a)^2+(y'-b)^2+(z'-c)^2\}-\{(x'-a)\,l+(y'-b)\,m+(z'-c)\,n\}^2$

or $\quad PN^2=\{(x'-a)^2+(y'-b)^2+(z'-c)^2\}\,\{l^2+m^2+n^2\}-\{(x'-a)\,l+(y'-b)\,m+(z'-c)\,n\}^2$

$=\Sigma\,\{(y'-b)\,n-(z'-c)\,m\}^2$ [by Lagrange's identity]

$\therefore\quad PN=\sqrt{[\Sigma\,\{(y'-b)\,n-(z'-c)\,m\}^2]}$.

Aliter. Let $\angle PAN=\theta$.

We have, $\quad PN^2=AP^2\sin^2\theta$.

Now θ is the angle between the lines AP and AB. Here

the d.c.'s of AP are $(x'-a)/AP,\ (y'-b)/AP,\ (z'-c)/AP$

and the d.c.'s of AB are $l,\ m,\ n$

$$\therefore\quad \sin^2\theta=\begin{vmatrix}(y'-b)/AP & (z'-c)/AP \\ m & n\end{vmatrix}^2+\begin{vmatrix}(x'-a)/AP & (z'-c)/AP \\ l & n\end{vmatrix}^2+\begin{vmatrix}(x'-a)/AP & (y'-b)/AP \\ l & m\end{vmatrix}^2$$

$$=\frac{1}{AP^2}\left[\begin{vmatrix}y'-b & z'-c \\ m & n\end{vmatrix}^2+\begin{vmatrix}x'-a & z'-c \\ l & n\end{vmatrix}^2+\begin{vmatrix}x'-a & y'-b \\ l & m\end{vmatrix}^2\right].$$

$$\therefore\quad PN^2=AP^2\sin^2\theta=\begin{vmatrix}y'-b & z'-c \\ m & n\end{vmatrix}^2+\begin{vmatrix}x'-a & z'-c \\ l & n\end{vmatrix}^2+\begin{vmatrix}x'-a & y'-b \\ l & m\end{vmatrix}^2. \quad ...(1)$$

Remark. In the formula (1), l, m, n are the d.c.'s of the line AB. If however, α, β, γ are the d.r.'s of the line AB, then to get PN^2 we should divide the R.H.S. of (1) by $\alpha^2+\beta^2+\gamma^2$.

SOLVED EXAMPLES 2 (C)

Ex. 1. *If points P, Q are (2, 3, −6), (3, −4, 5), then find the angle between OP and OQ where O is origin.*

Sol. The direction ratios of OP are $2-0,\ 3-0,\ -6-0$ *i.e.*, are $2, 3, -6$.

Also $OP=\sqrt{\{(2-0)^2+(3-0)^2+(-6-0)^2\}}=\sqrt{(49)}=7$.

$\therefore$ the d.c.'s of OP are $2/7,\ 3/7,\ -6/7$.

The direction ratios of OQ are $3-0,\ -4-0,\ 5-0$, *i.e.*, $3, -4, 5$.

Also $OQ=\sqrt{\{(3-0)^2+(-4-0)^2+(5-0)^2\}}=\sqrt{(50)}=5\sqrt{2}$.

$\therefore$ the d.c.'s of OQ are $3/(5\sqrt{2}),\ -4/(5\sqrt{2}),\ 5/(5\sqrt{2})$.

If θ is the angle between OP and OQ, then [See eqn. (5), § 12]

$$\cos\theta = l_1l_2+m_1m_2+n_1n_2.$$

$$\therefore \quad \cos\theta = \frac{2}{7}\cdot\frac{3}{5\sqrt{2}}+\frac{3}{7}\cdot\left(\frac{-4}{5\sqrt{2}}\right)+\left(-\frac{6}{7}\right)\cdot\left(\frac{5}{5\sqrt{2}}\right)=\frac{-36}{35\sqrt{2}}$$

or $$\cos\theta=\frac{-18\sqrt{2}}{35}. \quad \therefore \quad \theta=\cos^{-1}\left(\frac{-18\sqrt{2}}{35}\right).$$ **Ans.**

Ex. 2 (a). *If points P, Q are* (2, 3, 4), (1, −2, 1), *then prove that OP is perpendicular to OQ where O is* (0, 0, 0). **[Magadh 68]**

Sol. The direction ratios of OP are $2-0, 3-0, 4-0$, *i.e.* 2, 3, 4.

The direction ratios of OQ are $1-0, -2-0, 1-0$ *i.e.* 1, −2, 1.

If OP is perpendicular to OQ, then we must have

$$a_1a_2+b_1b_2+c_1c_2=0. \qquad \text{[See eqn. (13), § 12]}$$

Now $a_1a_2+b_1b_2+c_1c_2=(2).(1)+(3).(-2)+(4).(1)$
$=2-6+4=0,$

which shows that OP is perpendicular to OQ.

Ex. 2 (b). *Show that the line joining the points* (0, 1, 2) *and* (3, 4, 6) *is parallel to the line joining the points* (−4, 3, −6) (5, 12, 6).

Sol. The direction ratios of the line joining (0, 1, 2) and (3, 4, 6) are $3-0, 4-1, 6-2$ *i.e.*, 3, 3, 4.

The direction ratios of the line joining (−4, 3, −6) and (5, 12, 6) are $5-(-4), 12-3, 6-(-6)$ *i.e* , 9, 9, 12.

We see that the direction ratios of the two lines are proportional because we have $3/9=3/9=4/12$.

Hence the two lines are parallel.

Ex. 3. *If the vertices P, Q and R of a triangle have coordinates* (2, 3, 5), (−1, 3, 2) *and* (3, 5, −2) *respectively, find the angles of the triangle PQR.*

Sol. The direction ratios of PQ are $-1-2, 3-3, 2-5$ *i.e*, −3, 0, −3.

We have $\sqrt{\{(-3)^2+0^2+(-3)^2\}}=\sqrt{(18)}=3\sqrt{2}$.

$\therefore$ the d.c.'s of PQ are $\frac{-3}{3\sqrt{2}}, \frac{0}{3\sqrt{2}}, \frac{-3}{3\sqrt{2}}$, *i.e.*, $-\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$.

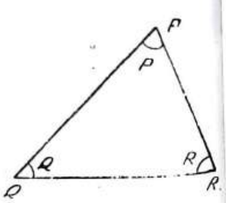
Similarly d.c.'s of QR are

$$\frac{4}{\sqrt{(36)}}, \frac{2}{\sqrt{(36)}}, \frac{-4}{\sqrt{(36)}}$$

i.e. $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$

and d.c.'s of PR are

$$\frac{1}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}}.$$



Now using the formula [eqn. 5, § 12] $\cos\theta=l_1l_2+m_1m_2+n_1n_2$, we shall find the angles P, Q, R of $\triangle PQR$ as follows :

We have, $\cos P=$ cos of angle between PQ and PR

$$=-\frac{1}{\sqrt{2}}\cdot\frac{1}{3\sqrt{6}}+0.\frac{2}{3\sqrt{6}}+\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{-7}{3\sqrt{6}}\right)=\frac{6}{3\sqrt{(12)}}=\frac{6}{6\sqrt{3}}$$

$=1/\sqrt{3}$. $\therefore$ $\angle P=\cos^{-1}(1/\sqrt{3})$.

Again $\cos Q=$ cos of angle between QP and QR

$$=\frac{1}{\sqrt{2}}\cdot\frac{2}{3}+0+\frac{1}{\sqrt{2}}\left(-\frac{2}{3}\right)=0. \quad \therefore \quad \angle Q=90°.$$

$$\left[\text{Note that d.c.'s of } QP \text{ are } \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right]$$

Finally $\cos R=$ cos of angle between RP and RQ

$$=\left(-\frac{2}{3}\right)\left(-\frac{1}{3\sqrt{6}}\right)+\left(-\frac{1}{3}\right)\left(-\frac{2}{3\sqrt{6}}\right)+\left(\frac{2}{3}\right)\left(\frac{7}{3\sqrt{6}}\right)$$

$$=\frac{18}{9\sqrt{6}}=\sqrt{\left(\frac{2}{3}\right)}.$$

$\therefore$ $R=\cos^{-1}\sqrt{(2/3)}$.

Ex. 4. *Prove that the straight lines whose direction cosines are given by the relations* $al+bm+cn=0$ *and* $fmn+gnl+hlm=0$ *are perpendicular if* $f/a+g/b+h/c=0$ *and parallel if* $\sqrt{(af)}\pm\sqrt{(bg)}\pm\sqrt{(ch)}=0$.

(Meerut 1984, 85 P, 87, 89)

Sol. From the first relation, we have $n=-(al+bm)/c$.

Putting this value of n in the second relation, we have

$$fm\left(-\frac{al+bm}{c}\right)+gl\left(-\frac{al+bm}{c}\right)+hlm=0$$

or $$afml+bfm^2+agl^2+bglm-chlm=0$$

or $$ag\frac{l^2}{m^2}+\frac{l}{m}(af+bg-ch)+bf=0. \quad ...(1)$$

Now if l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the two lines, then the roots of (1) are l_1/m_1 and l_2/m_2.

$\therefore$ product of the roots $=\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{bf}{ag}$ or $\frac{l_1l_2}{f/a}=\frac{m_1m_2}{g/b}$

$\therefore$ $\frac{l_1l_2}{(f/a)}=\frac{m_1m_2}{(g/b)}=\frac{n_1n_2}{(h/c)}$, by symmetry.

We know that the lines are perpendicular if

$$l_1l_2+m_1m_2+n_1n_2=0$$

i.e. if $f/a+g/b+h/c=0$, which proves the first part.

If the lines are parallel then the direction cosines are the same. This shows that the roots of (1) are equal, for which the condition is '$B^2=4AC$'

i.e. $$(af+bg-ch)^2=4ag.bf.$$

Taking square root, $af+bg-ch=\pm 2\sqrt{(afbg)}$

or $$af\pm 2\sqrt{(af\,bg)}+bg=ch$$

or $$\{\sqrt{(af)}\pm\sqrt{(bg)}\}^2=(ch).$$

Taking square root, $\sqrt{(af)}\pm\sqrt{(bg)}=\pm\sqrt{(ch)}$

or $$\sqrt{(af)}\pm\sqrt{(bg)}\pm\sqrt{(ch)}=0,$$ **Proved**

which proves the second result.

Ex. 5. *Show that the lines whose direction cosines are given by the equations $2l+2m-n=0$, and $mn+nl+lm=0$ are at right angles.* **(Meerut 1986 S)**

Sol. From $2l+2m-n=0$, we have $n=2l+2m$. ...(1)

Putting this value of n in $mn+nl+lm=0$, we get

$$m(2l+2m)+(2l+2m)\,l+lm=0$$

or $$2l^2+5lm+2m^2=0 \quad \text{or} \quad (l+2m)(2l+m)=0.$$

$\therefore$ $l=-2m$ and $2l+m=0$.

When $l=-2m$, from (1) $n=-2m$.

$$\therefore \quad \frac{l}{2}=\frac{m}{-1}=\frac{2}{n}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(2)^2+(-1)^2+(2)^2\}}}=\frac{1}{3}.$$

$\therefore$ The d.c.'s of one line are $2/3, -1/3, 2/3$.

Again when $2l=-m$, from (1), $n=m$.

$$\therefore \quad \frac{l}{-1}=\frac{m}{2}=\frac{n}{2}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(-1)^2+(2)^2+(2)^2\}}}=\frac{1}{3}$$

$\therefore$ The d.c.'s of other line are $-1/3, 2/3, 2/3$.

The lines will be at right angles if $l_1l_2+m_1m_2+n_1n_2=0$.

We have

$$l_1l_2+m_1m_2+n_1n_2=\frac{2}{3}\cdot\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=0.$$

Hence the lines are at right angles.

Ex. 6. *Show that the straight lines whose direction cosines are given by the relations $al+bm+cn=0$ and $ul^2+vm^2+wn^2=0$ are perpendicular or parallel according as*

$$a^2(v+w)+b^2(u+w)+c^2(u+v)=0$$

or $$a^2/u+b^3/v+c^2/w=0.$$ **(Meerut 1984, 86, 89S)**

Sol. From the first relation, we have $n=-(al+bm)/c$.

Putting this value of n in the second relation, we have

$$ul^2+vm^2+w\left\{-\frac{(al+bm)}{c}\right\}^2=0$$

or $$(c^2u+a^2w)\,l^2+2abwlm+(b^2w+c^2v)\,m^2=0$$

or $$(c^2u+a^2w)\,(l/m)^2+2abw\,(l/m)+(b^2w+c^2v)=0. \quad \text{...(1)}$$

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of the two lines.

Then the roots of (1) are l_1/m_1 and l_2/m_2.

$\therefore$ product of the roots$=\dfrac{l_1}{m_1}\cdot\dfrac{l_2}{m_2}=\dfrac{b^2w+c^2v}{c^2u+a^2w}$.

$$\therefore \quad \frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{a^2v+b^2u}, \text{ by symmetry.} \quad \text{...(2)}$$

The lines will be perpendicular if $l_1l_2+m_1m_2+n_1n_2=0$

i.e. $$(b^2w+c^2v)+(c^2u+a^2w)+(a^2v+b^2u)=0$$

or $$a^2(v+w)+b^2(u+w)+c^2(v+u)=0.$$ **Proved I.**

The lines will be parallel if the d.c.'s of the lines are the same *i.e.* if the roots of (1) are equal, for which the condition is

$$B^2=4AC,$$

i.e. $$4a^2b^2w^2=4\,(c^2u+a^2w)\,(b^2w+c^2v)$$

or $$a^2c^2wv+b^2c^2uw+c^2a^2vu=0,$$

or $$a^2/u+b^2/v+c^2/w=0.$$ **Proved II.**

Ex. 7. *Show that the lines whose d.c.'s are given by $l+m+n=0$ and $2mn+3ln-5lm=0$ are at right angles.* **(Meerut 1983)**

Sol. From the first relation, we have $l=-m-n$. ...(1)

Putting this value of l in the second relation, we have

$$2mn+3\,(-m-n)\,n-5\,(-m-n)\,m=0$$

or $$5m^2+2mn-3n^2=0 \quad \text{or} \quad 5\,(m/n)^2+2\,(m/n)-3=0. \quad \text{...(2)}$$

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of the two lines. Then the roots of (2) are m_1/n_1 and m_2/n_2.

$$\therefore \quad \text{product of the roots}=\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=-\frac{3}{5} \quad \text{or} \quad \frac{m_1m_2}{3}=\frac{n_1n_2}{-5}. \quad \text{...(3)}$$

Again from (1), $n=-l-m$ and putting this value of n in the second given relation, we have

$$2m\,(-l-m)+3l\,(-l-m)-5lm=0,$$

or $$3\,(l/m)^2+10\,(l/m)+2=0.$$

$$\therefore \quad \frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{2}{3} \quad \text{or} \quad \frac{l_1l_2}{2}=\frac{m_1m_2}{3} \quad \text{...(4)}$$

From (3) and (4), we have $\dfrac{l_1l_2}{2}=\dfrac{m_1m_2}{3}=\dfrac{n_1n_2}{-5}=k$ (say).

$\therefore$ $l_1l_2+m_1m_2+n_1n_2=(2+3-5)\,k=0.k=0$.

$\therefore$ The lines are at right angles. **Proved.**

Ex. 8. *Lines OA, OB are drawn from O with direction cosines proportional to 1, −2, −1; 3, −2, 3. Find the direction cosines of the normal to the plane AOB.*

Sol. Since OA and OB lie in the plane AOB, therefore these lines are perpendicular to any normal to the plane AOB.

Let a_1, a_2, a_3 be the direction ratios of such a normal, Using the condition of perpendicularity $a_1b_1+a_2b_2+a_3b_3=0$,

we have $a_1(1)+a_2(-2)+a_3(-1)=0$ ...(1)

and $a_1(3)+a_2(-2)+a_3(3)=0$. ...(2)

Solving (1) and (2), $\frac{a_1}{-2.3-(-1)(-2)}=\frac{a_2}{-1.3-1.3}=\frac{a_3}{1.(-2)-(-2)3}$

or $\frac{a_1}{-8}=\frac{a_2}{-6}=\frac{a_3}{4}$ or $\frac{a_1}{4}=\frac{a_2}{3}=\frac{a_3}{-2}$

Therefore the d.c.'s of the normal are

$$\frac{4}{\sqrt{\{(4)^2+(3)^2+(-2)^2\}}}, \frac{3}{\sqrt{\{(4)^2+(3)^2+(-2)^2\}}}, \frac{-2}{\sqrt{\{(4)^2+(3)^2+(-2)^2\}}}$$

or $\frac{4}{\sqrt{(29)}}, \frac{3}{\sqrt{(29)}}, \frac{-2}{\sqrt{(29)}}$.

Ex. 9. *If l_1, m_1, n_1 and l_2, m_2, n_2 are direction cosines of the two lines show that the direction cosines of the line perpendicular to both are proportional to $m_1n_2-m_2n_1, n_1l_2-n_2l_1, l_1m_2-l_2m_1$.*

Prove further if the given lines are at right angles to each other then these direction ratios are the actual direction cosines.

Sol. Suppose that the required direction cosines of the line are l, m, n. Since the line is perpendicular to the given lines, we have

$ll_1+mm_1+nn_1=0$ ...(1) and $ll_2+mm_2+nn_2=0$. ...(2)

Solving (1) and (2), we have

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}. \quad ...(3)$$

This shows that the required d.r.'s are $m_1n_2-m_2n_1, n_1l_2-n_2l_1, l_1m_2-l_2m_1$.

Now suppose θ is the angle between the two given lines whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2.

Then $\sin\theta=\sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}}$ [see cor. 1, § 12] ...(4)

If $\theta=90°$ *i.e.*, the lines are perpendicular, then (4) gives

$\sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}}=1$. ...(5)

$\therefore$ in this case from (3) the d.c.'s l, m, n of the line are given by

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}}}$$

$=\frac{1}{1}$ [Using (5) and $l^2+m^2+n^2=1$]

$=1$.

Hence in this case the actual direction cosines l, m, n are $m_1n_2-m_2n_1, n_1l_2-n_2l_1, l_1m_2-l_2m_1$. **Proved.**

Ex. 10. *Show that the direction cosines of a line perpendicular to a pair of mutually perpendicular lines with direction cosines as l_1, m_1, n_1 and l_2, m_2, n_2 respectively are*

$m_1n_2-m_2n_1, n_1l_2-n_2l_1, l_1m_2-l_2m_1$. **(Bhagalpur 1966)**

Sol. See example 9. This is other way of stating the same problem.

Ex. 11. *Prove that three concurrent lines with direction cosines l_1, m_1, n_1; l_2, m_2, n_2; and l_3, m_3, n_3 are coplanar if*

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0.$$

(Meerut 1983 S)

Sol. Let l, m, n be the d.c.'s of the normal to the plane in which the two concurrent lines with d.c.'s l_1, m_1, n_1 and l_2, m_2, n_2 lie. Then the line whose d.c.'s are l, m, n is perpendicular to the lines whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2. Therefore

$l_1l+m_1m+n_1n=0$ ...(1) and $l_2l+m_2m+n_2n=0$. ...(2)

Again if the third concurrent line whose d.c.'s are l_3, m_3, n_3 also lies in this plane, then it is also perpendicular to the normal to this plane.

$\therefore$ $l_3l+m_3m+n_3n=0$. ...(3)

Eliminating l, m, n between (1), (2) and (3), we get the required condition as

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0.$$

Ex. 12. *Prove that the three lines drawn from a point with direction cosines proportional to 1, −1, 1; 2, −3, 0, and 1, 0, 3 are coplanar.*

Sol. Let a, b, c be the direction ratios of the normal to the plane in which the two concurrent lines with direction ratios 1, −1, 1 and 2, −3, 0 lie.

Clearly these lines will be perpendicular to this normal. Hence applying the condition for perpendicularity of two lines, we have

$1.a+(-1).b+1.c=0$, ...(1)

and $$2.a+(-3).b+0.c=0. \quad ...(2)$$

Solving these, $\dfrac{a}{0+3}=\dfrac{b}{2-0}=\dfrac{c}{-3+2}$ or $\dfrac{a}{3}=\dfrac{b}{2}=\dfrac{c}{-1}$.

Again the third concurrent line with d.r.'s 1, 0, 3 will lie in this plane if the normal with d.r.'s 3, 2, -1 is also perpendicular to this third line.

We have (3) (1)+(2) (0)+(-1) (3)=0, showing that the lines with d.r.'s 3, 2, -1 and 1, 0, 3 are perpendicular. Hence the three given lines are coplanar.

Ex. 13. *The d.c.'s of two intersecting lines are l_1, m_1, n_1 and l_2, m_2, n_2. Show that all lines through the intersection of these two whose d.c.'s are proportional to $l_1+kl_2, m_1+km_2, n_1+kn_2$ are coplanar with them.*

Sol. Let l, m, n be the d.c.'s of the normal to the plane in which two intersecting lines whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2 lie. Clearly these lines will be perpendicular to this normal. Therefore $$l_1l+m_1m+n_1n=0, \quad ...(1)$$
and $$l_2l+m_2m+n_2n=0. \quad ...(2)$$

Now any line through the point of intersection of these lines with d.c.'s proportional to $l_1+kl_2, m_1+km_2, n_1+kn_2$ will lie in this plane if the normal to this plane is also perpendicular to this line.

We have $l(l_1+kl_2)+m(m_1+km_2)+n(n_1+kn_2)$
$$=ll_1+mm_1+nn_1+k(ll_2+mm_2+nn_2)$$
$$=0+k.0, \text{ using (1) and (2)}$$
$$=0.$$

This shows that the line whose d.c.'s are l, m, n is perpendicular to a line whose d.r.'s are $l_1+kl_2, m_1+km_2, n_1+kn_2$.

Hence all lines through the intersection of the given lines and with d.c.'s proportional to $l_1+kl_2, m_1+km_2, n_1+kn_2$ are coplanar with them.

Ex. 14. *If a variable line in two adjacent positions has direction cosines l, m, n and $l+\delta l, m+\delta m, n+\delta n$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2=(\delta l)^2+(\delta m)^2+(\delta n)^2$.*
(Meerut 1984 P, 87 P, 88)

Sol. Since l, m, n and $(l+\delta l), (m+\delta m), (n+\delta n)$ are the actual direction cosines, we have $l^2+m^2+n^2=1 \quad ...(1)$

and $(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1$

or $(l^2+m^2+n^2)+2l\delta l+2m\delta m+2n\delta n+(\delta l)^2+(\delta m)^2+(\delta n)^2=1$

or $1+2(l\delta l+m\delta m+n\delta n)+(\delta l)^2+(\delta m)^2+(\delta n)^2=1$ [Using (1)]

or $2(l\delta l+m\delta m+n\delta n)=-\{(\delta l)^2+(\delta m)^2+(\delta n)^2\}. \quad ...(2)$

Now it is given that $\delta\theta$ is the angle between two adjacent positions of the line. Therefore
$$\cos\delta\theta=l.(l+\delta l)+m.(m+\delta m)+n.(n+\delta n). \quad ...(3)$$

Now $\cos\delta\theta=1-\dfrac{(\delta\theta)^2}{2!}+\dfrac{(\delta\theta)^4}{4!}+...$

$\therefore$ if $\delta\theta$ is small, we have $\cos\delta\theta=1-\dfrac{(\delta\theta)^2}{2!}$.

Then from (3), we have
$$1-\frac{(\delta\theta)^2}{2!}=(l^2+m^2+n^2)+(l\delta l+m\delta m+n\delta n)$$

or $1-\dfrac{(\delta\theta)^2}{2}=1-\frac{1}{2}\{(\delta l)^2+(\delta m)^2+(\delta n^2)\}$ [using (1) and (2)]

or $(\delta\theta)^2=(\delta l)^2+(\delta m)^2+(\delta n)^2$. **Proved.**

Ex. 15. *Prove that the acute angle between the lines whose direction cosines are given by the relations*
$l+m+n=0$ and $l^2+m^2-n^2=0$ is $\pi/3$. **(Meerut 1986 P)**

Sol. The relations giving the d.c.'s of the two lines are
$l+m+n=0, \quad ...(1)$ and $l^2+m^2-n^2=0. \quad ...(2)$

From (1), $n=-(l+m)$. Putting this value of n in (2), we get
$l^2+m^2-(l+m)^2=0$, or $2lm=0$, or $lm=0$.
$\therefore \; l=0$ or $m=0$.

When $l=0$, we have from (1), $m=-n$.
$$\therefore \quad \frac{l}{0}=\frac{m}{1}=\frac{n}{-1}. \quad ...(3)$$

Again when $m=0$, we have from (1), $l=-n$.
$$\therefore \quad \frac{l}{1}=\frac{m}{0}=\frac{n}{-1}. \quad ...(4)$$

From (3) and (4), we observe that the d.r.'s of the two lines are 0, 1, -1; and 1, 0, -1.

$\therefore$ their d.c.'s are 0, $1/\sqrt{2}$, $-1/\sqrt{2}$; and $1/\sqrt{2}$, 0, $-1/\sqrt{2}$.

If θ is the acute angle between the two lines, we have
$$\cos\theta=\left|\left(0.\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right).0+\left(-\frac{1}{\sqrt{2}}\right).\left(-\frac{1}{\sqrt{2}}\right)\right|=\frac{1}{2}.$$
$\therefore \; \theta=\pi/3$

Ex. 16. *Show that the area of a triangle whose vertices are the origin and the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is*
$$\tfrac{1}{2}\sqrt{\{(y_1z_2-y_2z_1)^2+(z_1x_2-z_2x_1)^2+(x_1y_2-x_2y_1)^2\}}.$$

Sol. The direction ratios of OA are x_1, y_1, z_1 and those of OB are x_2, y_2, z_2.

Also $OA=\sqrt{\{(x_1-0)^2+(y_1-0)^2+(z_1-0)^2\}}=\sqrt{(x_1^2+y_1^2+z_1^2)}$

and $OB=\sqrt{\{(x_2-0)^2+(y_2-0)^2+(z_2-0)^2\}}=\sqrt{(x_2^2+y_2^2+z_2^2)}$.

$\therefore$ the d c.'s of OA are

$$\frac{x_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}, \frac{y_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}, \frac{z_1}{\sqrt{(x_1^2+y_1^2+z_1^2)}}$$

and the d.c.'s of OB are

$$\frac{x_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}, \frac{y_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}, \frac{z_2}{\sqrt{(x_2^2+y_2^2+z_2^2)}}.$$

Hence if θ is the angle between the lines OA and OB, then

$$\sin\theta=\frac{\sqrt{\{\Sigma(y_1z_2-y_2z_1)^2\}}}{\sqrt{(x_1^2+y_1^2+z_1^2)}\sqrt{(x_2^2+y_2^2+z_2^2)}}=\frac{\sqrt{\{\Sigma(y_1z_2-y_2z_1)^2\}}}{OA.OB}.$$

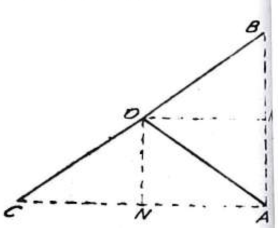
Hence the area of $\triangle OAB=\frac{1}{2}.OA.OB\sin\theta$ $[\because \angle AOB=\theta]$

$$=\tfrac{1}{2}.OA.OB.\frac{\sqrt{\{\Sigma(y_1z_2-z_1y_2)^2\}}}{OA.OB}$$

$$=\tfrac{1}{2}\sqrt{\{\Sigma(y_1z_2-y_2z_1)^2\}}.$$ **Proved.**

Ex. 17. *If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c.'s of two concurrent lines, show that the d.c.'s of two lines bisecting the angles between them are proportional to $l_1\pm l_2, m_1\pm m_2, n_1\pm n_2$.*

Sol. Let O be the origin. Draw OA and OB parallel to the given concurrent lines. Let l_1, m_1, n_1 be the d.c.'s of OA and l_2, m_2, n_2 the d.c.'s of OB. Cut off $OA=OB=r$. Then the co-ordinates of A and B are (l_1r, m_1r, n_1r) and (l_2r, m_2r, n_2r) respectively.



Again take a point C on BO produced such that $OB=OC=r$. Thus the co-ordinates of C are $(-l_2r, -m_2r, -n_2r)$.

Suppose M and N are the middle points of AB and CA. Then the co-ordinates of M and N are

$$\left(\frac{l_1r+l_2r}{2}, \frac{m_1r+m_2r}{2}, \frac{n_1r+n_2r}{2}\right)$$

and $$\left(\frac{l_1r-l_2r}{2}, \frac{m_1r-m_2r}{2}, \frac{n_1r-n_2r}{2}\right)$$

respectively. Clearly OM and ON are the internal and external bisectors of the angle AOB.

Hence the direction ratios of OM and ON are

$\frac{1}{2}(l_1+l_2)r, \frac{1}{2}(m_1+m_2)r, \frac{1}{2}(n_1+n_2)r$

and $\frac{1}{2}(l_1-l_2)r, \frac{1}{2}(m_1-m_2)r, \frac{1}{2}(n_1-n_2)r$ respectively,

i.e., the d.c.'s of OM and ON are proportional to $l_1+l_2, m_1+m_2, n_1+n_2$ and $l_1-l_2, m_1-m_2, n_1-n_2$ respectively. **Proved.**

Ex. 18. *Find the direction cosines of the lines bisecting the angles between the lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 and the angle between these lines is θ.*

Sol. Proceeding as in Ex. 17 above the direction ratios of the internal bisector OM of the angle AOB are

$$l_1+l_2, m_1+m_2, n_1+n_2. \quad ...(1)$$

Since θ is the angle between the lines whose d c.'s are l_1, m_1, n_1 and l_2, m_2, n_2,

$\therefore \cos\theta=l_1l_2+m_1m_2+n_1n_2=\Sigma l_1l_2$.

Also $\Sigma l_1^2=l_1^2+m_1^2+n_1^2=1$ and $\Sigma l_2^2=l_2^2+m_2^2+n_2^2=1$.

We have

$$\sqrt{\{(l_1+l_2)^2+(m_1+m_2)^2+(n_1+n_2)^2\}}=\sqrt{\{\Sigma l_1^2+\Sigma l_2^2+2\Sigma l_1l_2\}}$$
$$=\sqrt{(1+1+2\cos\theta)}=\sqrt{\{2(1+\cos\theta)\}}=2\cos\tfrac{1}{2}\theta.$$

Dividing the direction ratios (1) by

$$\sqrt{\{(l_1+l_2)^2+(m_1+m_2)^2+(n_1+n_2)^2\}}$$

i.e. by $2\cos\theta/2$, the d.c.'s of the internal bisector OM are

$$\frac{l_1+l_2}{2\cos\theta/2}, \frac{m_1+m_2}{2\cos\theta/2}, \frac{n_1+n_2}{2\cos\theta/2}.$$ **Ans.**

Similarly the d.c.'s of the external bisector ON are

$$\frac{l_1-l_2}{2\sin\theta/2}, \frac{m_1-m_2}{2\sin\theta/2}, \frac{n_1-n_2}{2\sin\theta/2}.$$ **Ans.**

Note that in this case, we have

$$\sqrt{\{(l_1-l_2)^2+(m_1-m_2)^2+(n_1-n_2)^2\}}=\sqrt{\{\Sigma l_1^2+\Sigma l_2^2-2\Sigma l_1l_2\}}$$
$$=\sqrt{(1+1-2\cos\theta)}=2\sin\tfrac{1}{2}\theta.$$

Ex. 19. *The vertices of a triangle PQR are the points $(-1, 2, -3)$, $(5, 0, -6)$ and $(0, 4, -1)$ in order. Find the direction ratios of the bisectors of the angle QPR.*

Sol. Suppose the internal bisector of the $\angle QPR$ meets the side QR in L. Then we know that

$$QL : LR=PQ : PR. \quad ...(1)$$

Now $PQ=\sqrt{\{(5+1)^2+(0-2)^2+(-6+3)^2\}}=\sqrt{(49)}=7$,

and $PR=\sqrt{\{(0+1)^2+(4-2)^2+(-1+3)^2\}}=\sqrt{(9)}=3$.

Putting the values of PQ and PR in (1), we have

$$QL : LR=7 : 3.$$

This shows that L divides QR **internally** in the ratio 7 : 3 and hence the co-ordinates of L are

$$\left(\frac{7.0+3.5}{7+3}, \frac{7.4+3.0}{7+3}, \frac{7.(-1)+3.(-6)}{7+3}\right)$$

i.e., $$\left(\frac{3}{2}, \frac{14}{5}, -\frac{5}{2}\right).$$

$\therefore$ the direction ratios of the internal bisector PL are

$$\left(\frac{3}{2}+1, \frac{14}{5}-2, -\frac{5}{2}+3\right) \text{ i.e., are } \frac{5}{2}, \frac{4}{5}, \frac{1}{2} \text{ i.e., are } 25, 8, 5.$$

Again the **external bisector** of the $\angle QPR$ will meet the side QR in M, where M divides QR externally in the ratio 7 : 3 and hence the co-ordinates of M are

$$\left(\frac{7.0-3.5}{7-3}, \frac{7.4-3.0}{7-3}, \frac{7.(-1)-3.(-6)}{7-3}\right)$$

i.e., $$\left(-\frac{15}{4}, 7, \frac{11}{4}\right).$$

$\therefore$ the direction ratios of the external bisector PM are

$$\left(-\frac{15}{4}+1, 7-2, \frac{11}{4}+3\right) \text{ i.e., are } -\frac{11}{4}, 5, \frac{23}{4}$$

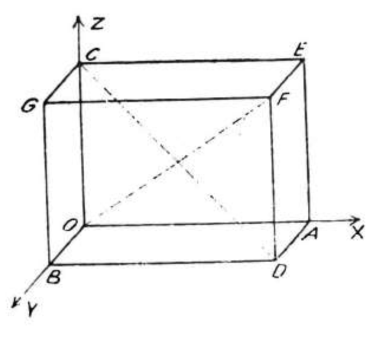
i.e., are $-11, 20, 23$.

Ex. 20. *If the edges of a rectangular parallelopiped be a, b, c show that the angles between the four diagonals are given by*

$$\cos^{-1}\left\{\frac{\pm a^2 \pm b^2 \pm c^2}{(a^2+b^2+c^2)}\right\}.$$

(Kanpur 1982, Meerut 84 S)

Sol. Let O, one of the vertices of the rectangular parallelopiped, be taken as origin and the three coterminous edges OA, OB and OC as the co-ordinate axes.



Let the edges $OA=a$, $OB=b$, $OC=c$. The four diagonals are OF, AG, BE and DC.

The co-ordinates of the vertices of the parallelopiped are given by O (0, 0, 0) ; A (a, 0, 0) ; B (0, b, 0) ; C (0, 0, c) ; F (a, b, c) ; D (a, b, 0) ; E (a, 0, c) ; G (0, b, c).

Note that the vertex D lies in the xy-plane, the vertex E in the zx-plane and the vertex G in the yz-plane.

The d.r.'s of the diagonal OF are $a-0, b-0, c-0$ *i.e.*, a, b, c.

$\therefore$ the d.c.'s of the diagonal OF are

$$\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \frac{b}{\sqrt{(a^2+b^2+c^2)}}, \frac{c}{\sqrt{(a^2+b^2+c^2)}}.$$

Similarly the d.c.'s of the diagonal AG are

$$\frac{-a}{\sqrt{(a^2+b^2+c^2)}}, \frac{b}{\sqrt{(a^2+b^2+c^2)}}, \frac{c}{\sqrt{(a^2+b^2+c^2)}};$$

the d.c.'s of the diagonal BE are

$$\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \frac{-b}{\sqrt{(a^2+b^2+c^2)}}, \frac{c}{\sqrt{(a^2+b^2+c^2)}};$$

and the d.c.'s of the diagonal CD are

$$\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \frac{b}{\sqrt{(a^2+b^2+c^2)}}, \frac{-c}{\sqrt{(a^2+b^2+c^2)}}.$$

$\therefore$ the angle θ between the diagonals OF and AG is given by

$$\cos\theta = \frac{a\times(-a)+b\times b+c\times c}{\sqrt{(a^2+b^2+c^2)}\sqrt{(a^2+b^2+c^2)}}$$

[using $\cos\theta = l_1l_2+m_1m_2+n_1n_2$]

i.e., $$\theta = \cos^{-1}\left\{\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right\}.$$

The total number of pairs of the diagonals are ${}^4C_2$ *i.e.*, 6. In a similar way the angles between the rest five pairs of the diagonals are determined and all of these six angles are given by

$$\cos^{-1}\left\{\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right\}.$$

The above expression will give only six valid values because the ambiguous signs cannot be either all +ive or −ive for in that case $\theta = \cos^{-1} 1$ or $\cos^{-1}(-1)$ *i.e.*, $\theta = 0$ or 180° which is impossible as no two of the diagonals are parallel.

Ex. 21. *A line makes angles α, β, γ, δ with the four diagonals of a cube. Prove that* $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = 4/3$.

(Gorakhpur 1982; Meerut 81, 83, 85; Lucknow 81; Kanpur 83)

Sol. In a cube all the edges are equal. Proceeding as above in **Ex.** 20 the d.c.'s of the diagonal OF (putting $b=c=a$) are

$$\frac{a}{\sqrt{(a^2+a^2+a^2)}}, \frac{a}{\sqrt{(a^2+a^2+a^2)}}, \frac{a}{\sqrt{(a^2+a^2+a^2)}}$$

i.e., $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$.

Similarly the d.c.'s of the diagonal AG are

$$-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3},$$

the d.c.'s of the diagonal BE are $1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}$,

and the d.c.'s of the diagonal CD are $1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}$.

Now let l, m, n be the d.c.'s of the line which makes angle $\alpha, \beta, \gamma, \delta$ with the four diagonals of the cube. Then we have

$$\cos\alpha = l.(1/\sqrt{3})+m.(1/\sqrt{3})+n.(1/\sqrt{3})=(l+m+n)/\sqrt{3}$$

or $\cos^2\alpha=(l+m+n)^2/3$, ...(1)

$$\cos\beta=l.(-1/\sqrt{3})+m.(1/\sqrt{3})+n.(1/\sqrt{3})=(-l+m+n)/\sqrt{3},$$

or $\cos^2\beta=(-l+m+n)^2/3$. ...(2)

Similarly $\cos^2\gamma=(l-m+n)^2/3$ ...(3)

and $\cos^2\delta=(l+m-n)^2/3$. ...(4)

Adding the relations (1), (2), (3) and (4), we get

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$$
$$=\tfrac{1}{3}\{(l+m+n)^2+(-l+m+n)^2+(l-m+n)^2+(l+m-n)^2\}$$
$$=\tfrac{1}{3}\{4(l^2+m^2+n^2)\}=\tfrac{4}{3}. \qquad [\because l^2+m^2+n^2=1]$$

Ex. 22. *Find the angle between two diagonals of a cube.*

Sol. Proceeding as in Ex. 21 above, the d.c.'s of the two diagonals OF and AG of the cube are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ and $-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ respectively. Thus if θ is the acute angle between the diagonals OF and AG, then we have

$$\cos\theta=|(1/\sqrt{3}).(-1/\sqrt{3})+(1/\sqrt{3}).(1/\sqrt{3})+(1/\sqrt{3}).(1/\sqrt{3})|=\tfrac{1}{3}$$

i.e., $\theta=\cos^{-1}(\tfrac{1}{3})$.

This is the required angle.

Ex. 23. *If* (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) *are the direction cosines of three mutually perpendicular lines, then find the direction cosines of a line whose direction cosines are proportional to* $l_1+l_2+l_3$, $m_1+m_2+m_3$, $n_1+n_2+n_3$, *and prove that this line is equally inclined to the given lines.*

Sol. Since l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the d.c.'s of three mutually perpendicular lines, we have

$$\left.\begin{array}{r} l_1l_2+m_1m_2+n_1n_2=0,\ l_2l_3+m_2m_3+n_2n_3=0,\\ l_3l_1+m_3m_1+n_3n_1=0.\\ \text{Also } l_1^2+m_1^2+n_1^2=l_2^2+m_2^2+n_2^2=l_3^2+m_3^2+n_3^2=1.\end{array}\right\} \quad ...(1)$$

Now we have $\sqrt{[(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2]}$

$$=\sqrt{[(l_1^2+m_1^2+n_1^2)+(l_2^2+m_2^2+n_2^2)+(l_3^2+m_3^2+n_3^2)}$$
$$+2\{(l_1l_2+m_1m_2+n_1n_2)+(l_2l_3+m_2m_3+n_2n_3)$$
$$+(l_3l_1+m_3m_1+n_3n_1)\}]$$
$$=\sqrt{[1+1+1+2(0+0+0)]}, \qquad \text{[using the relations (1)]}$$
$$=\sqrt{3}.$$

Thus the required d.c.'s of the line are given by

$$\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}. \qquad ...(2)$$

Let θ be the angle between the lines whose d.c.'s are l_1, m_1, n_1 and those given by (2). Then we have

$$\cos\theta=l_1.(l_1+l_2+l_3)/\sqrt{3}+m_1.(m_1+m_2+m_3)/\sqrt{3}$$
$$+n_1.(n_1+n_2+n_3)/\sqrt{3}$$
$$=(1/\sqrt{3})[(l_1^2+m_1^2+n_1^2)+(l_1l_2+m_1m_2+n_1n_2)$$
$$+(l_3l_1+m_3m_1+n_3n_1)]$$
$$=(1/\sqrt{3})[1+0+0], \qquad \text{[using the relations (1)]}.$$
$$\therefore \theta=\cos^{-1}(1/\sqrt{3}).$$

Similarly the angle between each of the lines with d.c.'s l_2, m_2, n_2; l_3, m_3, n_3 and the line with d.c.'s given by (2) is

$$\cos^{-1}(1/\sqrt{3}).$$

Ex. 24. *If two pairs of opposite edges of a tetrahedron are perpendicular, then prove that the third pair is also perpendicular.*

Sol. Let $OABC$ be a tetrahedron. Let O be chosen as origin. Let the co-ordinates of the vertices A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

The d.r.'s of the edges OA, OB and OC are x_1, y_1, z_1; x_2, y_2, z_2; and x_3, y_3, z_3 respectively. Also the d.r.'s of the edges BC, CA and AB are $x_3-x_2, y_3-y_2, z_3-z_2$; $x_1-x_3, y_1-y_3, z_1-z_3$ and $x_2-x_1, y_2-y_1, z_2-z_1$ respectively.

Now suppose the edge OA is perpendicular to the opposite edge BC. Then using the condition for the perpendicularity of two lines, we have

$$x_1(x_3-x_2)+y_1(y_3-y_2)+z_1(z_3-z_2)=0. \qquad ...(1)$$

Also if the edge OB is perpendicular to the opposite edge CA, then we have

$$x_2(x_1-x_3)+y_2(y_1-y_3)+z_2(z_1-z_3)=0. \qquad ...(2)$$

Adding (1) and (2), we have

$$x_3(x_1-x_2)+y_3(y_1-y_2)+z_3(z_1-z_2)=0. \qquad ...(3)$$

The relation (3) shows that the third pair of opposite edges OC and AB is also perpendicular.

Ex. 25. *If a pair of opposite edges of a tetrahedron be perpendicular, then show that the distances between the middle points of the other two pairs of opposite edges are equal.*

Sol. Let $OABC$ be a tetrahedron. Proceeding as in Ex. 24 above, if the pair OA and BC of opposite edges be perpendicular, then we have

$$x_1(x_3-x_2)+y_1(y_3-y_2)+z_1(z_3-z_2)=0. \quad \ldots(1)$$

Let M_1 and M_2 be the middle points of OB and CA, so that their co-ordinates are given by $M_1(\frac{1}{2}x_2, \frac{1}{2}y_2, \frac{1}{2}z_2)$ and $M_2(\frac{1}{2}(x_1+x_3), \frac{1}{2}(y_1+y_3), \frac{1}{2}(z_1+z_3))$.

Now

$$M_1M_2^2=\{\tfrac{1}{2}(x_1+x_3-x_2)\}^2+\{\tfrac{1}{2}(y_1+y_3-y_2)\}^2+\{\tfrac{1}{2}(z_1+z_3-z_2)\}^2$$
$$=\tfrac{1}{4}\{(x_1+x_3-x_2)^2+(y_1+y_3-y_2)^2+(z_1+z_3-z_2)^2\}. \quad \ldots(2)$$

Similarly if M_3 and M_4 are the middle points of OC and AB, we have

$$M_3M_4^2=\tfrac{1}{4}\{(x_1+x_2-x_3)^2+(y_1+y_2-y_3)^2+(z_1+z_2-z_3)^2\}. \quad \ldots(3)$$

Now we want to prove that $M_1M_2=M_3M_4$.

Subtracting (3) from (2), we get

$$M_1M_2^2-M_3M_4^2$$
$$=\tfrac{1}{4}[\{(x_1+x_3-x_2)^2-(x_1+x_2-x_3)^2\}+\{(y_1+y_3-y_2)^2-(y_1+y_2-y_3)^2\}+\{(z_1+z_3-z_2)^2-(z_1+z_2-z_3)^2\}]$$
$$=\tfrac{1}{4}[2x_1(2x_3-2x_2)+2y_1(2y_3-2y_2)+2z_1(2z_3-2z_2)],$$

using the formula $a^2-b^2=(a+b)(a-b)$

$$=x_1(x_3-x_2)+y_1(y_3-y_2)+z_1(z_3-z_2)$$
$$=0, \text{ using (1).}$$

$\therefore \quad M_1M_2^2=M_3M_4^2$ or $M_1M_2=M_3M_4$.

Ex. 26. *If in a tetrahedron $OABC$, $OA^2+BC^2=OB^2+CA^2=OC^2+AB^2$, then show that its pairs of opposite edges are at right angles.*

Sol. $OABC$ is a tetrahedron. Let O be chosen as origin and let the co-ordinates of the vertices A, B and C be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

Now

$$OA^2+BC^2=\{(x_1-0)^2+(y_1-0)^2+(z_1-0)^2\}+\{(x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2\}$$
$$=(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)+(x_3^2+y_3^2+z_3^2)-2(x_2x_3+y_2y_3+z_2z_3)$$
$$=\Sigma x_1^2+\Sigma x_2^2+\Sigma x_3^2-2\Sigma x_2x_3. \quad \ldots(1)$$

Proceeding similarly, we have

$$OB^2+CA^2=\Sigma x_1^2+\Sigma x_2^2+\Sigma x_3^2-2\Sigma x_3x_1 \quad \ldots(2)$$

and

$$OC^2+AB^2=\Sigma x_1^2+\Sigma x_2^2+\Sigma x_3^2-2\Sigma x_1x_2. \quad \ldots(3)$$

Now $OA^2+BC^2=OB^2+CA^2$ gives

$$\Sigma x_1^2+\Sigma x_2^2+\Sigma x_3^2-2\Sigma x_2x_3=\Sigma x_1^2+\Sigma x_2^2+\Sigma x_3^2-2\Sigma x_3x_1$$

or $$\Sigma x_2x_3-\Sigma x_3x_1=0$$

or $$(x_2x_3+y_2y_3+z_2z_3)-(x_3x_1+y_3y_1+z_3z_1)=0$$

or $$x_3(x_2-x_1)+y_3(y_2-y_1)+z_3(z_2-z_1)=0.$$

This shows that the edge OC is perpendicular to the opposite edge AB *i.e.*, an opposite pair of edges of the tetrahedron is perpendicular.

Similarly by taking $OB^2+CA^2=OC^2+AB^2$ and $OA^2+BC^2=OC^2+AB^2$, the other two opposite pairs of edges of the tetrahedron will be at right angles.

EXERCISES

1. The direction cosines of two straight lines, inclined at an angle θ are l_1, m_1, n_1 and l_2, m_2, n_2. Show that direction cosines of the bisector of the angle between them are

$$\frac{l_1+l_2}{2\cos\theta/2}, \frac{m_1+m_2}{2\cos\theta/2}, \frac{n_1+n_2}{2\cos\theta/2}.$$

(Meerut 1984S, 85S)

2. Prove that the line joining the points (1, 2, 3) and (−1, −2, 3) is perpendicular to the line joining the points (−2, 1, 5) and (3, 3, 2). **(Meerut 1985 S)**

3

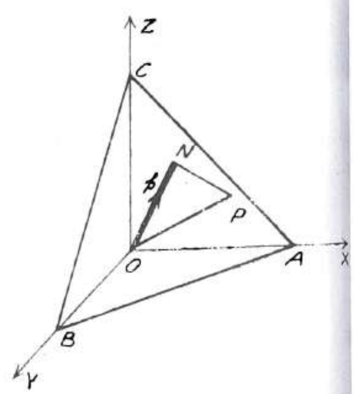
The Plane

§ 1. Plane.

Definition. *A plane is a surface such that all the points of a straight line joining any two points on the surface lie on it.* Or in other words *if we take any two points on the surface, the straight line joining these two points wholly lies on the surface.*

§ 2. (A) The equation of a plane. (Normal form).

To find the equation of a plane in terms of p i.e., the length of the perpendicular from the origin to the plane, and direction cosines l, m, n of this perpendicular.



Let OX, OY, OZ be a set of rectangular axes with O as origin. Let p be the length of the perpendicular ON from the origin O to the given plane ABC. We shall take p always positive. The d.c.'s of the perpendicular ON are l, m, n. The direction of the perpendicular ON is from origin towards the plane. A line perpendicular to the plane is called a *normal* to the plane. Thus l, m, n are the d.c.'s of the normal to the plane, the direction of the normal being that from the origin to the plane. If $\hat{\mathbf{n}}$ is a unit vector along the perpendicular ON, then

$$\hat{\mathbf{n}}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k} \text{ and hence } \vec{ON}=p\hat{\mathbf{n}}=p\,(l\mathbf{i}+m\mathbf{j}+n\mathbf{k}) \quad \ldots(1)$$

Let P with co-ordinates (x, y, z) be any point on the plane, so that

$$\vec{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}. \quad \ldots(2)$$

We have, $\vec{NP}=\vec{OP}-\vec{ON}=(x-pl)\,\mathbf{i}+(y-pm)\,\mathbf{j}+(z-pn)\,\mathbf{k}$.

So long as P lies on the plane, $\vec{NP}$ is always parallel to the plane and consequently perpendicular to $\vec{ON}$ and so also perpendicular to $\hat{\mathbf{n}}$ which is a unit vector in the direction of $\vec{ON}$.

$$\therefore \quad \vec{NP}\cdot\hat{\mathbf{n}}=0. \quad \ldots(3)$$

Putting the values of $\vec{NP}$ and $\hat{\mathbf{n}}$ in (3), we get

$$\{(x-pl)\,\mathbf{i}+(y-pm)\,\mathbf{j}+(z-pn)\,\mathbf{k}\}\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=0$$

or $$(x-pl)\,l+(y-pm)\,m+(z-pn)\,n=0,$$

or $$lx+my+nz-p\,(l^2+m^2+n^2)=0$$

or $$lx+my+nz=p \quad \ldots(4)$$

$$[\because \; l^2+m^2+n^2=1]$$

The equation (4) is satisfied by the co-ordinates of every point on the plane, but by no point [illegible] plane. Hence this is the equation of the plane [illegible] is known as the ***normal form*** of the equation of the plane.

Remark. In the equation (4), p is positive and $l^2+m^2+n^2=1$. Only then p is the length of the perpendicular from origin to the plane and l, m, n are the d.c.'s of the normal to the plane, the direction of the normal being from origin towards the plane.

Cor. If the perpendicular ON makes angles α, β, γ with the co-ordinate axes then clearly we have $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$, and the equation (4) becomes

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p. \quad \ldots(5)$$

(B) The general equation of a plane.

The equation (4) may be written as

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=p, \quad \ldots(6)$$

where l, m, n are the d.c.'s of the perpendicular ON.

Now suppose a, b, c are the d.r.'s of ON, so that l, m, n in terms of a, b, c are given by

$$(l, m, n)=(a, b, c)/\sqrt{(a^2+b^2+c^2)}.$$

Putting these values of l, m, n in the equation (6), we get

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})=p\sqrt{(a^2+b^2+c^2)}$$

or $$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})=q, \quad \ldots(7)$$

where $q=p\sqrt{(a^2+b^2+c^2)}$ and $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ is a vector perpendicular to the plane. *Hence the equation of the plane can be written in the form (7).*

Taking $q=-d$, the equation (7) of the plane may be written as

$$ax+by+cz+d=0. \quad ...(8)$$

The equation (8) is the general equation of a plane, where the numbers a, b, c are the d.r.'s of the normal to the plane *i.e.*, line perpendicular to the plane and the length of the perpendicular from origin to the plane is $-d/\sqrt{(a^2+b^2+c^2)}$, the number d being negative.

§ 3. *To prove that the general equation of the first degree in x, y, z namely $ax+by+cz+d=0$ represents a plane and that the coefficients a, b, c of x, y, z in this equation are d.r.'s of the normal to this plane.*

The general equation of the first degree in x, y, z is given by

$$ax+by+cz+d=0. \quad ...(1)$$

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points on the surface (1), so that we have

$$ax_1+by_1+cz_1+d=0 \quad ...(2)$$

and

$$ax_2+by_2+cz_2+d=0. \quad ...(3)$$

Multiplying the relation (3) by 'μ' and adding to (2), we get

$$a(x_1+\mu x_2)+b(y_1+\mu y_2)+c(z_1+\mu z_2)+d(1+\mu)=0.$$

Dividing both the sides by $(1+\mu)$, we get

$$\frac{a(x_1+\mu x_2)}{1+\mu}+\frac{b(y_1+\mu y_2)}{1+\mu}+\frac{c(z_1+\mu z_2)}{1+\mu}+d=0. \quad ...(4)$$

The relation (4) shows that for every value of $\mu \neq -1$, the point $\left(\frac{x_1+\mu x_2}{1+\mu}, \frac{y_1+\mu y_2}{1+\mu}, \frac{z_1+\mu z_2}{1+\mu}\right)$ lies on the surface (1). But these are the general co-ordinates of a point which divides the join of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $\mu : 1$. Since μ may take any real value other than -1, every point of the straight line AB lies on the surface (1). Hence the equation (1) represents a plane.

Subtracting (2) from (3), we have

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0. \quad ...(5)$$

The relation (5) shows that the two lines whose d.r.'s are a, b, c and $x_2-x_1, y_2-y_1, z_2-z_1$ are perpendicular. But x_2-x_1,

y_2-y_1, z_2-z_1 are d.r.'s of the line AB which is any line in the plane (1). Therefore a line whose d.r.'s are a, b, c is perpendicular to every line lying in the plane (1) and so it is perpendicular to the plane (1). Hence a, b, c are d.r.'s of the normal to the plane (1).

Remark. In the general co-ordinates of a point on the line AB, we cannot have $\mu=-1$ because there can be no point on the line AB which divides it in the ratio $-1 : 1$ *i.e.*, which divides it externally in the ratio $1 : 1$.

Note. The number of arbitrary constants in the general equation of the plane.

The general equation of the plane is

$$ax+by+cz+d=0 \quad \text{or} \quad (a/d)x+(b/d)y+(c/d)z=-1.$$

This equation shows that there are three arbitrary constants namely a/d, b/d, c/d in the equation of a plane. Therefore the equation of a plane can be determined to satisfy the three conditions, each condition giving us the value of a constant.

§ 4. To reduce the general equation of the plane to the normal form.

The general equation of the plane is

$$ax+by+cz+d=0. \quad ...(1)$$

If l, m, n are the d.c.'s of the normal to the plane, then the equation of the plane in the normal form is

$$lx+my+nz=p. \quad ...(2)$$

If (1) and (2) represent the same plane, then

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{p}{-d}=\pm\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2+b^2+c^2)}}=\pm\frac{1}{\sqrt{(a^2+b^2+c^2)}},$$

where the same sign either +ive or −ive is to be chosen throughout.

$$\therefore \quad l=\pm a/\sqrt{(a^2+b^2+c^2)},\ m=\pm b/\sqrt{(a^2+b^2+c^2)},$$
$$n=\pm c/\sqrt{(a^2+b^2+c^2)}, \text{ and } p=\pm d/\sqrt{(a^2+b^2+c^2)}.$$

Substituting these values in (2), the normal form of the plane (1) is given by

$$\pm\frac{ax}{\sqrt{(a^2+b^2+c^2)}}\pm\frac{by}{\sqrt{(a^2+b^2+c^2)}}\pm\frac{cz}{\sqrt{(a^2+b^2+c^2)}}=\pm\frac{d}{\sqrt{(a^2+b^2+c^2)}}. \quad ...(3)$$

The sign in equation (3) is so chosen that p *i.e.*, $\pm d/\sqrt{(a^2+b^2+c^2)}$ is always positive.

Working rule to reduce the equation of a plane in normal form.

Transpose the constant term in the equation of the plane to the R.H.S. and adjust the equation in such a way that this constant term on the R.H.S. is positive. Now divide the equation by $\sqrt{(a^2+b^2+c^2)}$, where a, b, c are the coefficients of x, y, z in the equation of the plane. The resulting equation will be the equation of the plane in the normal form

$$lx+my+nz=p.$$

Here p will be the length of the perpendicular from the origin to the plane and l, m, n will be the d.c.'s of the normal to the plane.

§ 5. Intercepts form :

To find the equation to the plane in terms of the intercepts a, b, c which the plane cuts on the coordinates axes. **(Meerut 1985)**

Let the general equation of the plane be

$$Ax+By+Cz+D=0. \quad ...(1)$$

Since the plane (1) makes an intercept a on the x-axis, the point $(a, 0, 0)$ lies on the plane (1), so that we have

$$Aa+D=0 \quad i.e., \quad A=-D/a.$$

Similarly the points $(0, b, 0)$ and $(0, 0, c)$ lie on the plane (1), so that

$$Bb+D=0 \text{ and } Cc+D=0, \text{ giving } B=-D/b, C=-D/c.$$

Putting the values of A, B, C in (1), we get

$$(-D/a)\, x+(-D/b)\, y+(-D/c)\, z+D=0$$

or

$$x/a+y/b+z/c=1. \quad ...(2)$$

The equation (2) is the required equation of the plane in terms of the intercepts a, b and c made by the plane on the axes of x, y and z respectively.

§ 6. Plane through a given point and perpendicular to a given line.

To find the equation of a plane through a given point $A(x_1, y_1, z_1)$ and perpendicular to a line whose direction ratios are a, b, c.

Let (x, y, z) be the coordinates of any current point P on the plane. Since the plane passes through the point $A\,(x_1, y_1, z_1)$, the line AP lies in the plane.

The d.r.'s of the line AP are $x-x_1, y-y_1, z-z_1$. Also the d.r.'s of the normal to the plane *i.e.*, of a line perpendicular to the plane are a, b, c.

Now the normal to the plane is perpendicular to every line lying in the plane and therefore the lines whose d.r.'s are a, b, c and $x-x_1, y-y_1, z-z_1$ are perpendicular.

$$\therefore \quad a\,(x-x_1)+b\,(y-y_1)+c\,(z-z_1)=0,$$

which is the equation of the required plane.

Remark. The equation of any plane passing through the point (x_1, y_1, z_1) is

$$a\,(x-x_1)+b\,(y-y_1)+c\,(z-z_1)=0.$$

In this equation a, b, c are d.r.'s of normal to the plane.

As a particular case, the equation of any plane passing through the origin is $ax+by+cz=0$, in which the coefficients of x, y, z *i.e.*, a, b, c are d.r.'s of the normal to the plane.

§ 7. Equation of a plane through three points.

To find the equation of a plane which passes through three points whose co-ordinates are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). **(Kanpur 1983)**

Let the general equation of the plane be

$$ax+by+cz+d=0. \quad ...(1)$$

If the equation (1) of the plane passes through the given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), the coordinates of these points will satisfy the equation (1), so that we have

$$ax_1+by_1+cz_1+d=0, \quad ...(2)$$

$$ax_2+by_2+cz_2+d=0, \quad ...(3)$$

and

$$ax_3+by_3+cz_3+d=0. \quad ...(4)$$

Eliminating a, b, c and d from the above equation (1), (2), (3) and (4), the equation of the required plane is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = \quad ...(5)$$

Cor. *Condition for four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) to be coplanar.*

The equation of the plane passing through first three points is given by equation (5). If the fourth point namely (x_4, y_4, z_4) also lies on this plane, then the co-ordinates of this point will satisfy the equation (5), so that we have

$$\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0, \; i.e., \; \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0. \quad ...(6)$$

The condition (6) is the required condition for four given points to be coplanar.

Note. For solving numerical examples an easier method can however be followed as explained in Ex. 7 and Ex. 8 below.

Solved Examples 3(A)

Ex. 1. *Reduce the equation of the plane $x+2y-2z-9=0$ to the normal form and hence find the length of the perpendicular drawn form the origin to the given plane.*

Sol. The equation of the given plane is

$$x+2y-2z-9=0.$$

Bringing the constant term to the R.H.S., the equation becomes $x+2y-2z=9$. ...(1)

[Note that in the equation (1) the constant term 9 is positive. If it were negative, we would have changed the sign throughout to moke it positive.]

Now the square root of the sum of the squares of the coefficients of x, y, z in (1) $=\sqrt{\{(1)^2+(2)^2+(-2)^2\}}=\sqrt{9}=3$.

Dividing both sides of (1) by 3, we have

$$\frac{1}{3}x+\frac{2}{3}y-\frac{2}{3}z=3. \quad ...(2)$$

The equation (2) of the plane is in the normal form

$$lx+my+nz=p.$$

Hence the d c 's l, m, n of the normal to the plane are $\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}$ and the length p of the perpendicular from the origin to the plane is 3.

Ex. 2. *The co-ordinates of a point A are (2, 3, −5). Determine the equation to the plane through A at right angles to the line OA, where O is the origin.* **(Meerut 1986S)**

Sol. Here the plane passes through the point A (2, 3, −5) and is perpendicular to the line OA i e, the line OA is normal to the plane.

The d.r.'s of OA are $2-0, 3-0, -5-0$ *i.e.*, 2, 3, −5.

Thus the plane passes through the point (2, 3, −5) and 2, 3, −5 are d.r.'s of normal to the plane. Therefore the equation of the plane is

$$2(x-2)+3(y-3)-5\{z-(-5)\}=0 \quad \text{[Refer § (6)]}$$

or $$2(x-2)+3(y-3)-5(z+5)=0$$

or $$2x+3y-5z-38=0$$

or $$2x+3y-5z=38.$$

Ex. 3. *O is the origin and A is the point (a, b, c). Find the d.c.'s of the join OA and deduce the equation of the plane through A at right angles to OA.*

Sol. The co-ordinates of the points O and A are (0, 0, 0) and (a, b, c) respectively. Thus the direction ratios of the join OA are $a-0, b-0, c-0$ *i.e.*, a, b, c.

Hence the d.c.'s of the join OA are

$a/\sqrt{(a^2+b^2+c^2)}, b/\sqrt{(a^2+b^2+c^2)}, c/\sqrt{(a^2+b^2+c^2)}$.

Now we are to find the equation of the plane passing through the point A (a, b, c) and perpendicular to the line OA. Here the line OA is normal to the plane and its d.r.'s are a, b, c. Therefore the equation of the required plane is

$a(x-a)+b(y-b)+c(z-c)=0$. or $ax+by+cz=a^2+b^2+c^2$.

Ex. 4. *Find the equation of the plane perpendicular to the line segment from (−3, 3, 2) to (9, 5, 4) at the middle point of the segment.*

Sol. The end points of the given line segment are (−3, 3, 2) and (9, 5, 4). The d.r.'s of the line segment are $9-(-3), 5-3, 4-2$ *i.e.*, 12, 2, 2. The co-ordinates of the middle point of the line segment are $(\frac{1}{2}(9-3), \frac{1}{2}(5+3), \frac{1}{2}(4+2))$ *i.e.*, (3, 4, 3).

Thus the plane is to pass through the point (3, 4, 3) and d.r.'s of normal to the plane are 12, 2, 2. Therefore the required equation of the plane is

$$12(x-3)+2(y-4)+2(z-3)=0$$

or $$6(x-3)+(y-4)+(z-3)=0, \text{ or } 6x+y+z=25.$$

Ex. 5. *Find the intercepts made on the co-ordinate axes by the plane* $x-3y+2z=9$.

Sol. The equation of the given plane is

$$x-3y+2z=9. \quad ...(1)$$

Dividing both sides by 9, the equation (1) may be written as

$$\frac{x}{9}+\frac{y}{-3}+\frac{z}{(9/2)}=1. \quad ...(2)$$

Comparing the equation (2) with the equation

$$x/a+y/b+z/c=1,$$

the intercepts on the co-ordinate axes are given by

a = the intercept on the x-axis = 9

b = the intercept on the y-axis = −3

and c = the intercept on the z-axis = 9/2.

Ex. 6. *A plane meets the co-ordinate axes in A, B, C such that the centroid of the triangle ABC is the point (p, q, r), show that the equation of the plane is* $x/p+y/q+z/r=3$.

(Kanpur 1983: Meerut 85S)

Sol. Let the equation of the plane be

$$x/a+y/b+z/c=1. \qquad ...(1)$$

The plane (1) meets the x-axis in the point A, so putting $y=0$ and $z=0$ in (1) we get $x=a$. Thus the co-ordinates of the point A are $(a, 0, 0)$. Similarly the plane (1) meets y and z axes in the points B and C whose co-ordinates are given by $B(0, b, 0)$ and $C(0, 0, c)$.

Thus the co-ordinates of the centroid of the triangle ABC are given by $(\frac{1}{3}(a+0+0), \frac{1}{3}(0+b+0), \frac{1}{3}(0+0+c))$ *i.e.*, $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$.

But it is given that the co-ordinates of the centroid of the triangle ABC are (p, q, r), so that we have

$$p=\tfrac{1}{3}a,\ q=\tfrac{1}{3}b,\ r=\tfrac{1}{3}c \quad \text{or} \quad a=3p,\ b=3q,\ c=3r.$$

Substituting these values of a, b, c in the equation (1), the equation of the required plane is given by

$$x/(3p)+y/(3q)+z/(3r)=1 \quad \text{or} \quad x/p+y/q+z/r=3.$$

Ex. 7. *Find the equation to the plane through the three points* $(0, -1, -1)$, $(4, 5, 1)$ *and* $(3, 9, 4)$.

Sol. The equation of any plane passing through the point $(0, -1, -1)$ is given by

$$a(x-0)+b\{y-(-1)\}+c\{z-(-1)\}=0$$

or

$$ax+b(y+1)+c(z+1)=0. \qquad ...(1)$$

If the plane (1) passes through the point (4, 5, 1), we have

$$4a+6b+2c=0. \qquad ...(2)$$

If the plane (1) passes through the point (3, 9, 4), we have

$$3a+10b+5c=0. \qquad ...(3)$$

Now solving the equations (2) and (3), we have

$$\frac{a}{30-20}=\frac{b}{6-20}=\frac{c}{40-18}=\lambda \text{ (say)}.$$

$$\therefore \quad a=10\lambda,\ b=-14\lambda,\ c=22\lambda.$$

Putting these values of a, b, c in (1), the equation of the required plane is given by

$$\lambda[10x-14(y+1)+22(z+1)]=0$$

or $$10x-14(y+1)+22(z+1)=0$$

or $$5x-7y+11z+4=0.$$

Ex. 8. *Show that the four points* $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ *and* $(-4, 4, 4)$ *are coplanar.* **(Meerut 1982)**

Sol. Proceeding as in Ex. 7 above the equation of the plane passing through the three points $(0, -1, -1)$, $(4, 5, 1)$ and $(3, 9, 4)$ is given by $5x-7y+11z+4=0$. ...(1)

If the fourth point $(-4, 4, 4)$ also lies on the plane (1) then the co-ordinates of this point must satisfy the equation (1).

Putting $x=-4$, $y=4$, $z=4$, the L.H.S. of (1)

$$=5.(-4)-7.4+11.4+4=0=\text{the R.H.S. of (1)}.$$

Hence the equation (1) is satisfied by the point $(-4, 4, 4)$. Therefore the given four points are coplanar.

§ 8. Equations of the co-ordinate planes.

(i) The equation to yz-plane. The x-coordinate of each point lying on the yz-plane is zero, and hence the equation to yz-plane is given by $x=0$.

(ii) The equation to zx-plane. It is giνen by $y=0$.

(iii) The equation to xy-plane. It is given by $z=0$.

§ 9. (A) The equations to the planes parallel to the co-ordinate planes.

The equation of the plane parallel to the yz-plane and at a distauce 'a' from it. The x-coordinate of every point on this plane is equal to 'a'. Hence the equation of the required plane is given by $x=a$.

Similarly, *the equation of the plane parallel to the* **xz**-*plane and at a distance 'b' from it is given by* $y=b$.

Also the equation of the plane parallel to the xy-plane and at a distance 'c' from it is given by $z=c$.

(B) The equation of the planes perpendicular to the co-ordinate axes.

The equation of the plane perpendicular to the x-axis. This plane is obviously parallel to the yz-plane and hence its equation is given by $x=a$. [See § 9(A) above].

Similarly the equations of the planes perpendicular to y and z axis are respectively given by $y=b$ and $z=c$.

§ 10. Angle between two planes.

Definition. *The angle between two planes is defined as the angle between their normols drawn from any point to the planes.*

Let the equations to the two planes be

$$a_1x+b_1y+c_1z+d_1=0. \qquad ...(1)$$

and

$$a_2x+b_2y+c_2z+d_2=0. \qquad ...(2)$$

The d.r.'s of the normal to the plane (1) are a_1, b_1, c_1 and the d.r.'s of the normal to the plane (2) are a_2, b_2, c_2.

If θ is the angle between the planes (1) and (2), then θ is the angle between the lines whose d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2.

$$\therefore \quad \cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}} \quad \ldots(3)$$

For the acute angle between the two planes, $\cos\theta$ is positive and for the obtuse angle it is negative. The numerical value of $\cos\theta$ in both these cases is the same because $\cos(\pi-\theta)=-\cos\theta$.

Condition of perpendicularity of two planes.

Two planes are perpendicular if their normals are perpendicular. Therefore the planes (1) and (2) are perpendicular if the lines whose d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular the condition for which is

$$a_1a_2+b_1b_2+c_1c_2=0. \quad \ldots(4)$$

Condition of parallelism of two planes.

Two planes are parallel if their normals are parallel. Therefore the planes (1) and (2) are parallel if the lines whose d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2 are parallel the condition for which is

$$a_1/a_2=b_1/b_2=c_1/c_2 \quad \ldots(5)$$

i.e., the coefficients of x, y, z in the equations of the two-planes should be proportional.

Remember. The equation of any plane parallel to the plane

$$ax+by+cz+d=0$$

is

$$ax+by+cz+\lambda=0.$$

§ 11. The two sides of a plane.

Two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same or opposite sides of the plane $ax+by+cz+d=0$ according as

$$ax_1+by_1+cz_1+d \text{ and } ax_2+by_2+cz_2+d$$

are of the same or opposite signs.

The equation of the plane is $\quad ax+by+cz+d=0 \quad \ldots(1)$

Suppose the line PQ meets the given plane (1) at the point R such that $PR : RQ = m_1 : m_2$. Then the co-ordinates of the point R are

$$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right).$$

Since the point R lies on the plane (1), therefore

$$a\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}\right)+b\left(\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)+c\left(\frac{m_1z_2+m_2z_1}{m_1+m_2}\right)+d=0$$

or

$$m_1(ax_2+by_2+cz_2+d)+m_2(ax_1+by_1+cz_1+d)=0$$

or

$$\frac{m_1}{m_2}=-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}. \quad \ldots(2)$$

Now the ratio m_1/m_2 is positive or negative according as PQ is divided at R internally or externally *i.e.*, the points P, Q lie on the opposite or same side of the plane (1).

Hence from (2) if $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of the same sign, then m_1/m_2 is negative *i.e.*, the points P and Q lie on the same side of the plane (1). If $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of the opposite signs, m_1/m_2 is positive *i.e.*, the points P and Q lie on the opposite sides of the plane (1).

§ 12. To find the length of the perpendicular from the point (x_1, y_1, z_1) to a given plane.

Let the equation of the given plane be

$$ax+by+cz=d=0. \quad \ldots(1)$$

To find the length of the perpendicular from the point (x_1, y_1, z_1) to the plane (1).

Shifting the origin to the point (x_1, y_1, z_1), the equation (1) becomes

$$a(x+x_1)+b(y+y_1)+c(z+z_1)+d=0$$

or

$$ax+by+cz+ax_1+by_1+cz_1+d=0. \quad \ldots(2)$$

Dividing both sides of (2) by $\sqrt{(a^2+b^2+c^2)}$, we get

$$\frac{a}{\sqrt{(a^2+b^2+c^2)}}x+\frac{b}{\sqrt{(a^2+b^2+c^2)}}y+\frac{c}{\sqrt{(a^2+b^2+c^2)}}z+\frac{ax_1+by_1+cz_1+d}{\sqrt{(a^2+b^2+c^2)}}=0. \quad \ldots(3)$$

The equation (3) of the plane is in the normal form with a proper adjustment of sign throughout the equation.

$\therefore$ The length of the perpendicular from the new origin to the plane (3) $\quad =\pm\dfrac{ax_1+by_1+cz_1+d}{\sqrt{(a^2+b^2+c^2)}}$.

Hence the length of the perpendicular from the point (x_1, y_1, z_1) to the plane (1) $\quad =\pm\dfrac{ax_1+by_1+cz_1+d}{\sqrt{(a^2+b^2+c^2)}}$.

Since the perpendicular distance of a point from the plane is always positive, therefore a positive or negative sign is to be attached before the radical according as $ax_1+by_1+cz_1+d$ is positive or negative *i.e.*, according as (x_1, y_1, z_1) lies on the same side or on opposite side of the plane as the origin, provided d is positive.

Working rule. To find the length p of the perpendicular from the point (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$, we substitute the co-ordinates of the given point in the left hand side of the equation of the plane and then divide this expression by

$$\sqrt{[(\text{coeff. of } x)^2+(\text{coeff of } y)^2+(\text{coeff. of } z)^2]}.$$

Thus
$$p=\frac{ax_1+by_1+cz_1+d}{\sqrt{(a^2+b^2+c^2)}}.$$

If the value of p obtained from this formula is negative, we can ignore the sign and give the positive value in answer, unless there is some special reason.

If the equation of the plane is in the normal form $lx+my+nz-p=0$, the length p_1 of the perpendicular from the point (x_1, y_1, z_1) to the plane is given by

$p_1=lx_1+my_1+nz_1-p$, for in this case $\sqrt{(l^2+m^2+n^2)}=1$.

§ 13. To find the distance between two parallel planes.

Find the lengths of perpendicular distances of each plane from the origin and retain their signs. The algebraic difference of these two perpendicular distances is the distance between the given parallel planes. But while applying this method we should be careful that the coefficients of x in the two equations of the planes are of the same sign.

Alternative method. Take a point on one of the two given planes, then the required distance is the length of the perpendicular drawn from this point to the other plane.

SOLVED EXAMPLES 3 (B)

Ex. 1. *Write the equations of the planes in the following cases:*

(*i*) *parallel to the xy-plane and* 5 *units below it,*

(*ii*) *parallel to the yz-plane and having x intercept* 3,

(*iii*) *perpendicular to the z-axis at the point* (0, 0, 4),

and (*iv*) *parallel to the zx-plane and* 6 *units behind it.*

Sol. In view of § 8, the equations of the planes in the different cases are as given below ;

(i) $z=-5$, (ii) $x=3$, (iii) $z=4$,

and (iv) $y=-6$, or $y+6=0$.

Ex. 2. *Find the equation of the plane which is horizontal and passes through the point* $(1, -2, -5)$.

Sol. Let us choose the co-ordinate axes so that the axes of x and y lie in a horizontal plane (the plane of the paper, say) and the axis of z is perpendicular to this plane. Hence the required plane is perpendicular to the z-axis and passes through

the point $(1, -2, -5)$, therefore, the equation of the required plane is given by $z=-5$ or $z+5=0$.

Remark. If the axes of x and z be taken in a horizontal plane (the plane of the paper, say) then y-axis is perpendicular to this plane and hence the equation of the required horizontal plane will be $y=-2$.

Ex. 3. *Find the equation of the plane through the point* $(-1, 2, 4)$ *and parallel to the plane* $2x+3y-5z+6=0$.

Sol. The equation of the given plane is

$$2x+3y-5z+6=0. \quad \text{...(1)}$$

Since the equations of the parallel planes differ only in the constant term, therefore the equation of any plane parallel to the plane (1) is given by

$$2x+3y-5z+k=0. \quad \text{...(2)}$$

If the plane (2) passes through the point $(-1, 2, 4)$, we have

$$2(-1)+3(2)-5(4)+k=0 \text{ or } k=16.$$

Substituting this value of k in the equation (2), the equation of the required plane is $2x+3y-5z+16=0$.

Ex. 4. *Find the equations of the planes parallel to the plane* $3x-6y-2z-4=0$ *at a distance* 3 *from the origin.*

Sol The equation of any plane parallel to the plane $3x-6y-2z-4=0$ is $3x-6y-2z+k=0$. ...(1)

Let p be the length of the perpendicular from the origin to the plane (1). Then

$$p=\pm\frac{k}{\sqrt{\{(3)^2+(-6)^2+(-2)^2\}}} \quad \text{[See § 12]}$$

or
$$p=\pm k/7.$$

But according to the given condition p is 3. Hence

$$k/7=\pm 3 \Rightarrow k=\pm 21.$$

Putting the values of k in (1), the equations of the required planes are given by $3x-6x-2z\pm 21=0$.

Ex. 5. *Find the equation of the plane parallel to the plane* $2x-3y-5z+1=0$ *and distant* 5 *units from the point* $(-1, 3, 1)$.

Sol. The equation of any plane parallel to the plane $2x-3y-5z+1=0$ is $2x-3y-5z+k=0$. ...(1)

Let q be the length of the perpendicular from the point $(-1, 3, 1)$ to the plane (1), then [see § 12]

$$q=\pm\frac{2(-1)-3(3)-5(1)+k}{\sqrt{\{(2)^2+(-3)^2+(-5)^2\}}}=\pm\frac{-16+k}{\sqrt{(38)}}.$$

But according to the question, $q=5$.

$\therefore \quad 5=\pm(-16+k)/\sqrt{(38)} \quad$ or $\quad k=16\pm5\sqrt{(38)}$.

Substituting the values of k in succession in (1), the equations of the required planes are given by

$$2x-3y-5z+16\pm5\sqrt{(38)}=0.$$

Ex. 6. *Find the equation of the plane through the point (α, β, γ) and parallel to the plane $ax+by+cz=0$.*

Sol. The equation of any plane parallel to the plane $ax+by+cz=0$ is $ax+by+cz+k=0$. ...(1)

If (1) passes through the point (α, β, γ), we have

$$a\alpha+b\beta+c\gamma+k=0$$

or $\quad k=-a\alpha-b\beta-c\gamma$.

Putting the value of k in the equation (1), the equation of the required plane is

$$ax+by+cz-a\alpha-b\beta-c\gamma=0$$

or $\quad ax+by+cz=a\alpha+b\beta+c\gamma$.

Ex. 7. *Find the equation of the plane through $(1, 0, -2)$ and perpendicular to each of the planes*

$$2x+y-z-2=0 \text{ and } x-y-z-3=0.$$

(Gorakhpur 1978; Meerut 85S)

Sol. The equation of any plane through the point $(1, 0, -2)$ is $a(x-1)+b(y-0)+c(z+2)=0$. ...(1)

If the plane (1) is perpendicular to the planes $2x+y-z-2=0$ and $x-y-z-3=0$, we have [See § 10]

$a(2)+b(1)+c(-1)=0$ *i.e.*, $2a+b-c=0$, ...(2)

and $\quad a(1)+b(-1)+c(-1)=0$ *i.e.*, $a-b-c=0$. ...(3)

Adding the equations (2) and (3), we have $c=\frac{3}{2}a$.

Subtracting (3) from (2), we have $b=-\frac{1}{2}a$.

Putting the values of b and c in (1), the equation of the required plane is given by

$$a(x-1)-\tfrac{1}{2}ay+\tfrac{3}{2}a(z+2)=0$$

or $\quad 2x-2-y+3z+6=0$, or $2x-y+3z+4=0$.

Ex. 8. *Find the equation of the plane through the points $(1, -2, 2)$, $(-3, 1, -2)$ and perpendicular to the plane*

$$x+2y-3z=5.$$

(Gorakhpur 1980)

Sol. The equation of any plane passing through the point $(1, -2, 2)$ is

$$a(x-1)+b(y+2)+c(z-2)=0. \quad ...(1)$$

If the plane (1) passes through the point $(-3, 1, -2)$, we have

$$a(-3-1)+b(1+2)+c(-2-2)=0$$

or $\quad -4a+3b-4c=0$, or $4a-3b+4c=0$. ...(2)

If the plane (1) is perpendicular to the plane $x+2y-3z=5$, we have $a.1+b.2+c.(-3)=0$, or $a+2b-3c=0$. ...(3)

Solving the equations (2) and (3) for a, b, c, we have

$$\frac{a}{(-3)(-3)-2(4)}=\frac{b}{1\times4-4.(-3)}=\frac{c}{4\times2-1.(-3)}$$

or $\quad \frac{a}{1}=\frac{b}{16}=\frac{c}{11}=\lambda$, (say). $\quad \therefore \quad a=\lambda,\ b=16\lambda,\ c=11\lambda$.

Putting the values of a, b, c in (1), the equation of the required plane is

$$\lambda[(x-1)+16(y+2)+11(z-2)]=0,$$

or $\quad x+16y+11z+9=0$.

Ex. 9. *Find the angle between the planes $2x-y+z=6$ and $x+y+2z=7$.*

Sol. The d.r.'s of the normal to the plane $2x-y+z=6$ are $2, -1, 1$.

The d.r.'s of the normal to the plane $x+y+2z=7$ are $1, 1, 2$.

Now the angle between two planes is equal to the angle between their normals. Therefore if θ be the angle between the given planes, we have

$$\cos\theta=\frac{2.1+(-1).1+1.2}{\sqrt{\{(2)^2+(-1)^2+(1)^2\}}\sqrt{\{(1)^2+(1)^2+(2)^2\}}}$$

$$=3/6=\tfrac{1}{2}. \quad \therefore \quad \theta=\tfrac{1}{3}\pi.$$

Ex. 10. *Find whether the two points $(2, 0, 1)$ and $(3, -3, 4)$ lie on the same side or opposite sides of the plane $x-2y+z=6$.*

Sol. Taking all terms to the left hand side, the equation of the plane may be written as $\quad x-2y+z-6=0$ ...(1)

Substituting the co-ordinates of the point $(2, 0, 1)$, the value of the left hand side of the equation (1) of the plane

$$=2-0+1-6=-3.$$

Again substituting the co-ordinates of the point $(3, -3, 4)$, the value of the left hand side of the equation (1) of the plane

$$=3+6+4-6=7.$$

Since the values -3 and 7 are of opposite signs, the given two points lie on the opposite sides of the given plane. [See § 11]

Ex. 11. *Find the equation of the locus of a point P whose distance from the plane $6x-2y+3z+4=0$ is equal to its distance from the point $(-1, 1, 2)$.*

But according to the question, $q=5$.

$\therefore\quad 5=\pm(-16+k)/\sqrt{(38)}$ or $k=16\pm5\sqrt{(38)}$.

Substituting the values of k in succession in (1), the equations of the required planes are given by

$$2x-3y-5z+16\pm5\sqrt{(38)}=0.$$

Ex. 6. *Find the equation of the plane through the point (α, β, γ) and parallel to the plane $ax+by+cz=0$.*

Sol. The equation of any plane parallel to the plane

$ax+by+cz=0$ is $ax+by+cz+k=0$. ...(1)

If (1) passes through the point (α, β, γ), we have

$$a\alpha+b\beta+c\gamma+k=0$$

or $$k=-a\alpha-b\beta-c\gamma.$$

Putting the value of k in the equation (1), the equation of the required plane is

$$ax+by+cz-a\alpha-b\beta-c\gamma=0$$

or $$ax+by+cz=a\alpha+b\beta+c\gamma.$$

Ex. 7. *Find the equation of the plane through $(1, 0, -2)$ and perpendicular to each of the planes*

$$2x+y-z-2=0 \text{ and } x-y-z-3=0.$$

(Gorakhpur 1978; Meerut 85S)

Sol. The equation of any plane through the point $(1, 0, -2)$ is $a(x-1)+b(y-0)+c(z+2)=0$. ...(1)

If the plane (1) is perpendicular to the planes $2x+y-z-2=0$ and $x-y-z-3=0$, we have [See § 10]

$a(2)+b(1)+c(-1)=0$ *i.e.*, $2a+b-c=0$, ...(2)

and $a(1)+b(-1)+c(-1)=0$ *i.e.*, $a-b-c=0$. ...(3)

Adding the equations (2) and (3), we have $c=\tfrac{3}{2}a$.

Subtracting (3) from (2), we have $b=-\tfrac{1}{2}a$.

Putting the values of b and c in (1), the equation of the required plane is given by

$$a(x-1)-\tfrac{1}{2}ay+\tfrac{3}{2}a(z+2)=0$$

or $2x-2-y+3z+6=0$, or $2x-y+3z+4=0$.

Ex. 8. *Find the equation of the plane through the points $(1, -2, 2)$, $(-3, 1, -2)$ and perpendicular to the plane*

$$x+2y-3z=5.$$

(Gorakhpur 1980)

Sol. The equation of any plane passing through the point $(1, -2, 2)$ is

$$a(x-1)+b(y+2)+c(z-2)=0. \quad ...(1)$$

If the plane (1) passes through the point $(-3, 1, -2)$, we have

$$a(-3-1)+b(1+2)+c(-2-2)=0$$

or $-4a+3b-4c=0$, or $4a-3b+4c=0$. ...(2)

If the plane (1) is perpendicular to the plane $x+2y-3z=5$, we have $a.1+b.2+c.(-3)=0$, or $a+2b-3c=0$. ...(3)

Solving the equations (2) and (3) for a, b, c, we have

$$\frac{a}{(-3)(-3)-2(4)}=\frac{b}{1\times4-4.(-3)}=\frac{c}{4\times2-1.(-3)}$$

or $\dfrac{a}{1}=\dfrac{b}{16}=\dfrac{c}{11}=\lambda$, (say). $\therefore\quad a=\lambda,\ b=16\lambda,\ c=11\lambda$.

Putting the values of a, b, c in (1), the equation of the required plane is

$$\lambda[(x-1)+16(y+2)+11(z-2)]=0,$$

or $$x+16y+11z+9=0.$$

Ex. 9. *Find the angle between the planes $2x-y+z=6$ and $x+y+2z=7$.*

Sol. The d.r.'s of the normal to the plane $2x-y+z=6$ are $2, -1, 1$.

The d.r.'s of the normal to the plane $x+y+2z=7$ are $1, 1, 2$.

Now the angle between two planes is equal to the angle between their normals. Therefore if θ be the angle between the given planes, we have

$$\cos\theta=\frac{2.1+(-1).1+1.2}{\sqrt{\{(2)^2+(-1)^2+(1)^2\}}\sqrt{\{(1)^2+(1)^2+(2)^2\}}}$$

$$=3/6=\tfrac{1}{2}. \quad \therefore \quad \theta=\tfrac{1}{3}\pi.$$

Ex. 10. *Find whether the two points $(2, 0, 1)$ and $(3, -3, 4)$ lie on the same side or opposite sides of the plane $x-2y+z=6$.*

Sol. Taking all terms to the left hand side, the equation of the plane may be written as $x-2y+z-6=0$ ...(1)

Substituting the co-ordinates of the point $(2, 0, 1)$, the value of the left hand side of the equation (1) of the plane

$$=2-0+1-6=-3.$$

Again substituting the co-ordinates of the point $(3, -3, 4)$, the value of the left hand side of the equation (1) of the plane

$$=3+6+4-6=7.$$

Since the values -3 and 7 are of opposite signs, the given two points lie on the opposite sides of the given plane. [See § 11]

Ex. 11. *Find the equation of the locus of a point P whose distance from the plane $6x-2y+3z+4=0$ is equal to its distance from the point $(-1, 1, 2)$.*

Sol. Let the co-ordinates of a point P be (x_1, y_1, z_1). It is required to find the locus of the point P.

Let p be the perpendicular distance of the point $P\ (x_1, y_1, z_1)$ from the plane $6x-2y+3z+4=0$.

Then $p=\dfrac{6x_1-2y_1+3z_1+4}{\sqrt{\{36+4+9\}}}=\dfrac{6x_1-2y_1+3z_1+4}{7}$ ...(1)

Again let d be the distance of the point $P\ (x_1, y_1, z_1)$ from the point $(-1, 1, 2)$.

Then $d=\sqrt{\{(x_1+1)^2+(y_1-1)^2+(z_1-2)^2\}}$. ...(2)

According to the given condition, we have $p=d$ *i.e.*, $p^2=d^2$

i.e. $(6x_1-2y_1+3z_1+4)^2/49=(x_1+1)^2+(y_1-1)^2+(z_1-2)^2$

or $36x_1^2+4y_1^2+9z_1^2+16-24x_1y_1+36z_1x_1+48x_1-12y_1z_1$
$-16y_1+24z_1=49\ \{x_1^2+2x_1+1+y_1^2-2y_1+1+z_1^2-4z_1+4\}$

or $13x_1^2+45y_1^2+40z_1^2+24x_1y_1-36z_1x_1+12y_1z_1+50x_1-82y_1$
$-220z_1+278=0.$

Hence the locus of the point $P\ (x_1, y_1, z_1)$ is
$13x^2+45y^2+40z^2+24xy-36zx+12yz+50x-82y-220z+278=0.$

Ex. 12. *Find the perpendicular distance between the parallel planes* $2x-3y-6z-21=0$ *and* $2x-3y-6z+14=0$.

Sol. The equations of the given planes are
$2x-3y-6z-21=0,$ ...(1) and $2x-3y-6z+14=0.$ ...(2)

[Note that the coefficients of x in the equation (1) ond (2) are of the same sign].

Let p_1 and p_2 be the lengths of the perpendicular distances of the planes (1) and (2) from the origin. Then we have

$$p_1=\frac{2.0-3.0-6.0-21}{\sqrt{(4+9+36)}}=\frac{-21}{7}=-3,$$

and $$p_2=\frac{2.0-3.0-6.0+14}{\sqrt{(4+9+36)}}=\frac{14}{7}=2.$$

Hence the perpendicular distance between the given planes
$=p_2-p_1=2-(-3)=5.$

Alternative method. The co-ordinates of a point on the first plane $2x-3y-6z-21=0$ are $(0, -7, 0)$.

The perpendicular distance between the given planes
$=$ the perpendicular distance of the point $(0, -7, 0)$ from the second plane $2x-3y-6z+14=0$

$$=\frac{0+21-0+14}{\sqrt{(4+9+36)}}=\frac{35}{7}=5.$$

Ex. 13. *Find the locus of a point, the sum of the squares of whose distances from the planes*
$x+y+z=0,\ x-y=0,\ x+y-2z=0$ *is 7.*

Sol. Let $P(x_1, y_1, z_1)$ be a point whose locus is required.

Let $p_1=$ the distance of P from the plane $x+y+z=0$
$=(x_1+y_1+z_1)/\sqrt{(3)}$,
$p_2=$ the distance of P from the plane $x-y=0$
$=(x_1-y_1)/\sqrt{(2)}$,

and $p_3=$ the distance of P from the plane $x+y-2z=0$
$=(x_1+y_1-2z_1)/\sqrt{(6)}$.

Now according to the given condition, we have

$$p_1^2+p_2^2+p_3^2=7. \quad ...(1)$$

Substituting the values of p_1, p_2, p_3 in (1), we get
$\frac{1}{3}(x_1+y_1+z_1)^2+\frac{1}{2}(x_1-y_1)^2+\frac{1}{6}(x_1+y_1-2z_1)^2=7$

or $2\ (x_1^2+y_1^2+z_1^2+2x_1y_1+2z_1x_1+2y_1z_1)+3\ (x_1^2-y_1^2-2x_1y_1)$
$+(x_1^2+y_1^2+4z_1^2+2x_1y_1-4z_1x_1-4y_1z_1)=42$

or $6x_1^2+6y_1^2+6z_1^2=42$, or $x_1^2+y_1^2+z_1^2=7$.

$\therefore$ the required locus of $P\ (x_1, y_1, z_1)$ is $x^2+y^2+z^2=7$.

§ 14. A plane through the intersection of two given planes.

To find the equation of any plane through the line of intersection of the two given planes.

Let the equations of the two given planes be

$P\equiv a_1x+b_1y+c_1z+d_1=0,$ ...(1)

and $Q\equiv a_2x+b_2y+c_2z+d_2=0.$ ...(2)

Then the equation $P+\lambda Q=0$, *i.e.*, the equation

$$(a_1x+b_1y+c_1z+d_1)+\lambda\ (a_2x+b_2y+c_2z+d_2)=0 \quad ...(3)$$

is the required equation of any plane through the line of intersection of the planes (1) and (2).

First we observe that the equation (1) is of first degree in x, y and z and so it represents a plane. Again whatever λ may be if a point satisfies both the equations (1) and (2), it definitely satisfies the equation (3). Thus all the points of the line of intersection of the planes (1) and (2) also lie on the plane (3). Hence (3) is the equation of any plane passing through the line of intersection of the planes (1) and (2).

§ 15. *To find the condition that a line whose d.r.'s are l, m, n may be parallel or be perpendicular to a given plane.*

Let the equation of the given plane be

$$ax+by+cz+d=0. \quad ...(1)$$

Thus the d.r.'s of the normal to the plane (1) are a, b, c. The d.r.'s of the given line are l, m, n.

The line is parallel to the plane. If the given line is parallel to the line (1), it is perpendicular to the normal to the plane (1) the condition for which is

$$al + bm + cn = 0. \qquad \ldots(1)$$

The line is perpendicular to the plane. If the given line is perpendicular to the plane (1), it is parallel to the normal to the plane (1), the condition for which is $\quad a/l = b/m = c/n$

§ 16. The angle between a line and a plane.

Definition. *The angle between a line and a plane is defined to be the complement of the angle between the line and the normal to the plane.*

Clearly this angle can be determined by the methods explained earlier.

Solved Examples 3(C)

Ex. 1. *Find the equation of the plane passing through the line of intersection of the planes* $2x-7y+4z=3$, $3x-5y+4z+11=0$ *and the point* $(-2, 1, 3)$.

Sol. The equation of any plane through the line of intersection of the given planes is [See § 14]

$$(2x-7y+4z-3)+\lambda\,(3x-5y+4z+11)=0. \qquad \ldots(1)$$

If the plane (1) passes through the point $(-2, 1, 3)$, then substituting the co-ordinates of this point in the equation (1) we have

$$\{2(-2)-7(1)+4(3)-3\}+\lambda\,\{3(-2)-5(1)+4(3)+11\}=0$$

or $\quad (-2)+\lambda\,(12)=0, \quad$ or $\quad \lambda=1/6$.

Putting this value of λ in (1), the equation of the required plane is $(2x-7y+4z-3)+(1/6)\,(3x-5y+4z+11)=0$

or $\quad 15x-47y+28z=7$.

Ex. 2. *Find the equation of the plane through the line of intersection of the planes* $x+2y-3z-6=0$ *and* $4x+3y-2z+2=0$ *and passing through the origin.*

Sol. The equation of any plane through the line of intersection of the given planes is

$$(x+2y-3z-6)+\lambda\,(4x+3y-2z+2)=0. \qquad \ldots(1)$$

If the plane (1) passes through the origin *i.e.*, through the point (0, 0, 0), then we have

$(0+0-0-6)+\lambda\,(0+0-0+2)=0$, or $-6+2\lambda=0$, or $\lambda=3$.

Substituting this value of λ in the equation (1), the equation of the required plane is

$$(x+2y-3z-6)+3\,(4x+3y-2z+2)=0,$$

or $\quad 13x+11y-9z=0$.

Ex. 3. *Find the equation of the plane through the line of intersection of the planes* $x+2y+3z-4=0$ *and* $2x+y-z+5=0$ *and perpendicular to the plane* $5x+3y+6z+8=0$.

(Lucknow 1982, Meerut 84S)

Sol. The equation of any plane through the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ is

$$(x+2y+3z-4)+\lambda\,(2x+y-z+5)=0$$

or $\quad x\,(1+2\lambda)+y\,(2+\lambda)+z\,(3-\lambda)-(4-5\lambda)=0. \qquad \ldots(1)$

If the plane (1) is perpendicular to the plane $5x+3y+6z+8=0$, we have $\quad (1+2\lambda).5+(2+\lambda).3+(3-\lambda).6=0$

or $\quad 5+10\lambda+6+3\lambda+18-6\lambda=0, \quad$ or $\quad 7\lambda+29=0, \quad$ or $\quad \lambda=-29/7$.

Substituting the value of λ in (1), the equation of the required plane is

$$x\,(1-58/7)+y\,(2-29/7)+z\,(3+29/7)-(4+145/7)=0$$

or $\quad -51x-15y+50z-173=0$,

or $\quad 51x+15y-50z+173=0$.

Ex. 4. *Find the equation of the plane through the line of intersection of the planes* $ax+by+cz+d=0$ *and* $\alpha x+\beta y+\gamma z+\delta=0$ *and perpendicular to the xy-plane.*

Sol. The equation of any plane through the line of intersection of the planes $ax+by+cz+d=0$ and $\alpha x+\beta y+\gamma z+\delta=0$ is

$$(ax+by+cz+d)+\lambda\,(\alpha x+\beta y+\gamma z+\delta)=0 \qquad \ldots(1)$$

or $\quad x\,(a+\alpha\lambda)+y\,(b+\beta\lambda)+z\,(c+\gamma\lambda)+(d+\delta\lambda)=0. \qquad \ldots(2)$

Now the equation of the xy-plane is given by $\quad z=0$

or $\quad 0.x+0.y+1.z=0 \qquad \ldots(3)$

If the planes (2) and (3) are perpendicular to each other, we have

$$0.(a+\alpha\lambda)+0.(b+\beta\lambda)+1.(c+\gamma\lambda)=0, \quad \text{or} \quad \lambda=-c/\gamma.$$

Putting this value of λ in the equation (1), the equation of the required plane is

$$(ax+by+cz+d)-(c/\gamma)\,(\alpha x+\beta y+\gamma z+\delta)=0$$

or $\quad \gamma\,(ax+by+cz+d)-c\,(\alpha x+\beta y+\gamma z+\delta)=0$

or $\quad (a\gamma-c\alpha)\,x+(b\gamma-c\beta)\,y+(c\gamma-c\gamma)\,z+(d\gamma-c\delta)=0$

or $\quad (a\gamma-c\alpha)\,x+(b\gamma-c\beta)\,y+(d\gamma-c\delta)=0$.

Ex. 5. *Find the equation of the plane through the line of intersection of the planes* $ax+by+cz+d=0$ *and* $\alpha x+\beta y+\gamma z+\delta=0$ *and parallel to x-axis.*

Sol. The equation of any plane through the line of intersection of the given planes is

$$ax+by+cz+d+\lambda(\alpha x+\beta y+\gamma z+\delta)=0 \quad ...(1)$$

or $$x(a+\alpha\lambda)+y(b+\beta\lambda)+z(c+\gamma\lambda)+(d+\delta\lambda)=0. \quad ...(2)$$

Now the d.c.'s of the x-axis are 1,0,0 and the d.r.'s of the normal to the plane (2) are $a+\alpha\lambda$, $b+\beta\lambda$, $c+\gamma\lambda$. The plane (2) is parallel to the x-axis if the normal to the plane (2) is perpendicular to the x-axis, the condition for which is

$$1.(a+\alpha\lambda)+0.(b+\beta\lambda)+0.(c+\gamma\lambda)=0,$$

giving $$\lambda=-a/\alpha.$$

Putting this value of λ in the equation (1), the required equation of the plane is given by

$$\alpha(ax+by+cz+d)-a(\alpha x+\beta\lambda+\gamma z+\delta)=0$$

or $$(b\alpha-a\beta)y+(c\alpha-a\gamma)z+(d\alpha-a\delta)=0.$$

Ex. 6. *Find the equation of the plane through the point* (1, −2, 0) *and normal to the line joining the points* (2, 3, −2) *and* (1, −2, 4).

Sol. The d.r.'s of the line joining the points (2, 3, −2) and (1, −2, 4) are 1−2, −2−3, 4−(−2) *i.e.*, −1, −5, 6. These are the d.r.'s of the normal to the plane. Since the plane is to pass through the point (1, −2, 0), its equation is

$$-1(x-1)-5(y+2)+6(z-0)=0,$$

or $$-x-5\lambda+6z-9=0 \text{ or } x+5y-6z+9=0,$$

or $$x+5y-6z=-9.$$

Ex. 7. *Find the equation of the plane through the points* (1, −2, 4) *and* (3, −4, 5) *and parallal to the x-axis* (*i.e. perpedicular to the yz-plane*).

Sol. The equation of any plane through the point (1, −2, 4) is

$$a(x-1)+b(y+2)+2)+c(z-4)=0. \quad ...(1)$$

If the plane (1) also passes through the point (3, −4, 5), we have

$$a(3-1)+b(-4+2)+c(5-4)=0,$$

or $$2a-2b+c=0. \quad ...(2)$$

Now the plane (1) is to be parallel to the x-axis *i.e.* perpendicular to the xy-plane whose equation is

$$x=0 \quad i.e., \quad 1.x+0.y+0.z=0.$$

Hence we have

$$a.1+b.0+c.0=0, \quad \text{or} \quad a=0. \quad ...(3)$$

Putting the value of a from (3) in (2), we get $c=2b$.

Substituting the values of a and c in the equation (1), the equation of the required palane is

$$0+b(y+2)+2b(z-4)=0, \quad \text{or} \quad y+2z-6=0.$$

Some Important Solved Examp les

Ex. 1. *A variable plane at a constant distance p from the origin meets the axes in A, B and C. Through A, B, C planes are drawn parallel to the co-ordinate planes. Show that the locus of their point of intersection is* $x^{-2}+y^{-2}+z^{-2}=p^{-2}$. **(Meerut 1984 P)**

Sol. Let the equation of the variable plane be

$$x/a+y/b+z/c=1, \quad ...(1)$$

where a, b, c are variables.

The plane (1) meets the co-ordinate axes in the points A, B and C whose co-ordinates are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

It is given that the length of the perpendicular from (0, 0, 0) to the plane (1) is p.

$$\therefore \quad p=\frac{1}{\sqrt{\{(1/a)^2+(1/b)^2+(1/c)^2\}}} \text{ or } \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}. \quad ...(2)$$

Now we shall find the equation of the plane through the point $A(a, 0, 0)$, and parallel to the yz-plane.

The equation of the yz-plane is $x=0$.

Any plane parallel to the plane $x=0$ is given by $x=\lambda$.

If it passes through the point $A(a, 0, 0)$, we have $a=\lambda$.

Hence the equation of the plane through A and parallel to the yz-plane is $$x=a. \quad ...(3)$$

Similarly the equations of the planes through the points B and C and parallel respectively to the co-ordinate planes $y=0$ and $z=0$ are $$y=b \quad ...(4) \quad \text{and} \quad z=c. \quad ...(5)$$

The locus of the point of intersection of the planes (3), (4) and (5) is obtained by eliminating a, b, c between the equations (2), (3), (4) and (5).

Putting the values of a, b, c from (3), (4) and (5) in (2), the required locus is given by

$$1/p^2=1/x^2+1/y^2+1/z^2 \text{ or } p^{-2}=x^{-2}+y^{-2}+z^{-2}.$$

Ex. 2. *A variable plane passes through a fixed point* (α, β, γ) *and meets the axes of reference in A, B, C. Show that the locus of the point of intersection of the planes through A, B, C parallel to the co-ordinate planes is* $\alpha x^{-1}+\beta y^{-1}+\gamma z^{-1}=1$. **(Meerut 1984)**

Sol. Let the equation of the variable plane be

$$x/a+y/b+z/c=1, \quad ...(1)$$

where a, b, c are parameters *i.e.*, variables.

The plane (1) passes through the point (α, β, γ).

$\therefore \quad \alpha/a+\beta/b+\gamma/c=1.$ ...(2)

The plane (1) meets the co-ordinate axes in the points A, B and C whose co-ordinates are respectively given by $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. The equation of the planes through A, B and C and parallel to the co-ordinate planes are

$$x=a,\ y=b,\ z=c \text{ respectively.} \quad ...(3)$$

[See Ex. 1 above]

The locus of the point of intersection of these planes [given by the equations (3)] is obtained by eliminating the parameters a, b, c between the equations (2) and (3). Putting the values of a, b, c from (3) in (2), the required locus is given by

$$\alpha/x+\beta/y+\gamma/z=1 \text{ or } \alpha x^{-1}+\beta y^{-1}+\gamma z^{-1}=1.$$

Ex. 3. *A point P moves on the plane $x/a+y/b+z/c=1$ which is fixed. The plane through P perpendicular to OP meets the co-ordinate axes in A, B and C. The planes through A, B and C parallel to the yz, zx and xy-planes intersect in Q. Prove that if the axes be rectangular, the locus of Q is*

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}.$$ **(Kanpur 1983)**

Sol. The equation of the plane is

$$x/a+y/b+z/c=1. \quad ...(1)$$

Let the co-ordinates of the point P be (α, β, γ). Since the point $P(\alpha, \beta, \gamma)$ lies on the plane (1), we have

$$\alpha/a+\beta/b+\gamma/c=1. \quad ...(2)$$

The direction ratios of OP are $\alpha-0, \beta-0, \gamma-0$ *i.e.*, α, β, γ. Hence the equation of the plane passing through the point $P(\alpha, \beta, \gamma)$ and perpendicular to OP is

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0,$$

or $$\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2. \quad ...(3)$$

The plane (3) meets the axes in the points A, B and C whose co-ordinates are respectively given by

$$(\{\alpha^2+\beta^2+\gamma^2\}/\alpha, 0, 0),\ (0, \{\alpha^2+\beta^2+\gamma^2\}/\beta, 0)$$

and $$(0, 0, \{\alpha^2+\beta^2+\gamma^2\}/\gamma).$$

Again the equation of the plane through A and parallel to the yz-plane *i.e.*, the plane $x=0$ is

$$x=(\alpha^2+\beta^2+\gamma^2)/\alpha. \quad ...(4)$$

Similarly the equations of the other two planes are

$$y=(\alpha^2+\beta^2+\gamma^2)/\beta \quad ...(5) \quad \text{and} \quad z=(\alpha^2+\beta^2+\gamma^2)/\gamma. \quad ...(6)$$

Now Q is the point of intersection of the planes (4), (5) and (6). The locus of the point Q is obtained by eliminating α, β, γ between the equations (2), (4), (5) and (6).

From (4), (5) and (6), we have

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\beta^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}$$

$$=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}=\frac{1}{\alpha^2+\beta^2+\gamma^2} \quad ...(7)$$

and $$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{\alpha/a+\beta/b+\gamma/c}{(\alpha^2+\beta^2+\gamma^2)}=\frac{1}{\alpha^2+\beta^2+\gamma^2} \text{ [using (2)]} \quad ...(8)$$

From (7) and (8), the required locus of Q is given by

$$1/x^2+1/y^2+1/z^2=1/(ax)+1/(by)+1/(cz).$$

Ex. 4. *A variable plane is at a constant distance 3p from the origin and meets the axes in A, B and C. Prove that the locus of the centroid of the triangle ABC is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.*

(Meerut 1983, 85, 89S ; Lucknow 81)

Sol. Let the equation of the variable plane be

$$x/a+y/b+z/c=1. \quad ...(1)$$

It is given that the length of the perpendicular from the origin to the plane (1) is $3p$.

$$\therefore \quad 3p=\frac{1}{\sqrt{(1/a^2+1/b^2+1/c^2)}} \text{ or } \frac{1}{9p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}. \quad ...(2)$$

The plane (1) meets the coordinate axes in the points A, B and C whose co-ordinates are respectively given by $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Let (x, y, z) be the co-ordinates of the centroid of the triangle ABC. Then

$$x=(a+0+0)/3,\ y=(0+b+0)/3,\ z=(0+0+c)/3$$

i.e., $$x=\tfrac{1}{3}a,\ y=\tfrac{1}{3}b,\ z=\tfrac{1}{3}c.$$

$$\therefore \quad a=3x,\ b=3y,\ c=3z. \quad ...(3)$$

The locus of the centroid of the triangle ABC is obtained by eliminating a, b, c between the equations (2) and (3). Putting the values of a, b, c from (3) in (2), the required locus is given by

$$\frac{1}{9p^2}=\frac{1}{9x^2}+\frac{1}{9y^2}+\frac{1}{9z^2} \text{ or } x^{-2}+y^{-2}+z^{-2}=p^{-2}.$$

Ex. 5 (a). *A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the triangle ABC is*

$$x^{-2}+y^{-2}+z^{-2}=9p^{-2}.$$ **(Lucknow 1981)**

Sol. Proceed as in Ex. 4 above.

Ex. 5 (b). *A plane meets the co-ordinate axes at A, B, C su that the centroid of the triangle ABC is the point (a, b, c). Find t locus to the plane ABC.* **(Meerut 198**

Sol. Let the equation of the plane be $x/a'+y/b'+z/c'=1$ ...(

the plane (1) meets the co-ordinates axes, in A, B, C. Hence we have $A(a', 0, 0)$, $B(0, b', 0)$ and $C(0, 0, c')$.

Since centroid of $\triangle ABC$ is (a, b, c)

$$\therefore \quad a=\tfrac{1}{3}(a'+0+0),\ b=\tfrac{1}{3}(0+b'+0),\ c=\tfrac{1}{3}(0+0+c')$$

or $\quad a'=3a,\ b'=3b,\ c'=3c.$

Substituting values in (1), the equation of the required pla

is $\quad x/a+y/b+z/c=3.$

Ex. 6. *A variable plane is at a constant distance p from t origin O and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is*

$$x^{-2}+y^{-2}+z^{-2}=16p^{-2}.$$ **(Lucknow 1982; Meerut 8**

Sol. Let the equation of the variable plane be

$$x/a+y/b+z/c=1. \quad ...(1)$$

It is given that the length of the perpendicular from the origi O to the plane (1) is p.

$$\therefore \quad p=\frac{1}{\sqrt{(1/a^2+1/b^2+1/c^2)}} \text{ or } \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}. \quad ...(2)$$

The plane (1) meets the axes in the points A, B and C who co-ordinates are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. Let (x, y, z) be the co-ordinates of the centroid of the tetrahedro $OABC$. Then

$$x=(0+a+0+0)/4,\ y=(0+0+b+0)/4,\ z=(0+0+0+c)/4.$$

$$\therefore \quad a=4x,\ b=4y,\ c=4z. \quad ...(3)$$

The locus of the centroid of the tetrahedron $OABC$ is obtaine by eliminating a, b, c between the equations (2) and (3). Puttin the values of a, b, c from (3) in (2), the equation of the require locus is given by

$$\frac{1}{p^2}=\frac{1}{16x^2}+\frac{1}{16y^2}+\frac{1}{16z^2} \text{ or } x^{-2}+y^{-2}+z^{-2}=16p^{-2}.$$

Ex. 7. *Two systems of rectangular axes have the same origi If a plane cuts them at distances a, b, c and a', b', c', respective from the origin, show that*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$ **(Lucknow 1978 ; Kanpur 8**

Sol. Let O be the origin. Let OX, OY, OZ be one set o

rectangular axes and let the equation of the plane with respect to this set of axes be $\quad x/a+y/b+z/c=1.$ ...(1)

Let the second set of rectangular axes be chosen as $O\xi$, $O\eta$, $O\zeta$ and let the equation of the same plane with respect to this set of axes be $\quad \xi/a'+\eta/b'+\zeta/c'=1.$ ...(2)

Now we know that from a point outside the plane only one perpendicular can be drawn to the plane. The origin being the same for both the systems, the length of the perpendicular from the origin to the plane in both the cases will be the same. Hence we have

$$\frac{1}{\sqrt{\{(1/a)^2+(1/b^2)+(1/c^2)\}}}=\frac{1}{\sqrt{\{(1/a')^2+(1/b')^2+(1/c')^2\}}}$$

or $\quad 1/a^2+1/b^2+1/c^2=1/a'^2+1/b'^2+1/c'^2.$

Ex. 8. *A plane meets a set of three mutually perpendicular planes in the sides of a triangle whose angles are A, B and C respectively. Saow that the first plane makes with the other planes angles, the squares of whose cosines are*

cot B cot C, cot C cot A, cot A cot B.

Sol. Let the three mutually perpendicular planes be chosen as the co-ordinate planes, and let

$$x/a+y/b+z/c=1, \quad ...(1)$$

be the equation of any palne which meets the axes of x, y and z at the points A, B and C respectively.

Clearly the co-ordinates of the vertices A, B, C of the $\triangle ABC$ are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ respectively.

The direction ratios of the side AB are $a, -b, 0$ and the direction ratios of AC are $a, 0, -c$. Hence the angle A between the sides AB and AC is given by

$$\tan A=\frac{\sqrt{[\Sigma(b_1c_2-b_2c_1)^2]}}{a_1a_2+b_1b_2+c_1c_2} \quad \text{[See § 12, chapter 2]}$$

or $\quad \tan A=\dfrac{\sqrt{[(bc-0)^2+(0+ac)^2+(0+ab)^2]}}{a^2+0+0}.$

$$\therefore \quad \cot A=a^2/\sqrt{(b^2c^2+a^2c^2+a^2b^2)}.$$

Similarly $\quad \cot B=b^2/\sqrt{(b^2c^2+c^2a^2+a^2b^2)}$

and $\quad \cot C=c^2/\sqrt{(a^2b^2+b^2c^2+c^2a^2)}.$

Now suppose α is the angle between the plane (1) and one of the co-ordinate planes say the plane $x=0$. The d.r.'s of the normals to these planes are $1/a, 1/b, 1/c$ and $1, 0, 0$ respectively.

$$\because \quad \cos\alpha=\frac{(1/a).1+(1/b).0+(1/c).0}{\sqrt{(1/a^2+1/b^2+1/c^2)}.\sqrt{(1^2+0^2+0^2)}}$$

or $$\cos^2 \alpha = \frac{b^2c^2}{(b^2c^2+c^2a^2+a^2b^2)} = \cot B \cot C$$

Similarly if β and γ are the angles which the plane (1) makes with the coordinate planes $y=0$ and $z=0$ respectively, then we have $\cos^2 \beta = \cot C \cot A$ and $\cos^2 \gamma = \cot A \cot B$.

Ex. 9. *Find the equation of the plane which bisects the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ perpendicularly.*

Sol. The required plane passes through the middle point of the segment PQ and is perpendicular to PQ. The co-ordinates of the middle point of PQ are

$$\left(\tfrac{1}{2}(x_1+x_2), \tfrac{1}{2}(y_1+y_2), \tfrac{1}{2}(z_1+z_2)\right).$$

Also the direction ratios of PQ are $(x_1-x_2, y_1-y_2, z_1-z_2)$. Thus the direction ratios of the normal to the plane are $x_1-x_2, y_1-y_2, z_1-z_2$ and hence the equation of the required plane is given by

$$(x_1-x_2)\{x-\tfrac{1}{2}(x_1+x_2)\}+(y_1-y_2)\{y-\tfrac{1}{2}(y_1+y_2)\}+(z_1-z_2)\{z-\tfrac{1}{2}(z_1+z_2)\}=0$$

or $$x(x_1-x_2)+y(y_1-y_2)+z(z_1-z_2)=\tfrac{1}{2}\{(x_1^2-x_2^2)+(y_1^2-y_2^2)+(z_1^2-z_2^2)\}.$$

Ex 10. *From any point P are drawn PM and PN perpendiculars to zx and xy-planes. O is the origin and α, β, γ and δ are the angles which OP makes with the co-ordinate planes and with the plane OMN. Prove that if the co-ordinates of the point P are (a, b, c) then.*

(*i*) *the equation of the plane OMN is $x/a-y/b-z/c=0$*

(*ii*) $$\delta = \sin^{-1} \frac{abc}{\sqrt{(a^2+b^2+c^2)}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}}$$

and (*iii*) $\operatorname{cosec}^2 \delta = \operatorname{cosec}^2 \alpha + \operatorname{cosec}^2 \beta + \operatorname{cosec}^2 \gamma$.

Sol. The co-ordinates of the point P are given to be (a, b, c). The equation of the zx-plane is $y=0$. Now since PM is drawn perpendicular from P to the zx- plane, therefore M is the foot of the perpendicular from $P(a\ b\ c)$ to the zx-plane and hence the co-ordinates of the point M are $(a, 0, c)$. Similarly the co-ordinates of the foot N (lying in the xy. plane) are $(a, b, 0)$.

Now we shall find the equation of the plane OMN.

The equation of any plane through the origin $O(0, 0, 0)$ is

$$Ax+By+Cz=0. \quad \ldots(1)$$

If the plane (1) passes through the points $M(a, 0, c)$ and $N(a, b, 0)$, we have $A.a+B.0+C.c=0$ and $A.a+B.b+C.0=0$.

Solving these for A, B, C, we get $\dfrac{A}{-bc}=\dfrac{B}{ca}=\dfrac{C}{ab}$.

Putting the values of A, B, C in (1) the equation of the plane OMN is given by

$$bcx+cay+abz=0 \text{ or } x/a-y/b-z/c=0. \quad \ldots(2)$$

This proves the result (i).

To find the angle δ. Since δ is the angle between the line OP and the plane OMN, therefore $90°-\delta$ is the angle between the line OP and the normal to the plane OMN.

The direction ratios of the line OP are $a-0, b-0, c-0$ *i.e.* a, b, c. The direction ratios of the normal to the plane OMN are $1/a, -1/b, -1/c$. [See equation (2)]

Hence $$\cos(90°-\delta)=\frac{a(1/a)+b(-1/b)+c(-1/c)}{\sqrt{(a^2+b^2+c^2)}\sqrt{(1/a^2+1/b^2+1/c^2)}}$$
$$=\frac{abc}{\sqrt{(a^2+b^2+c^2)}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}},$$

neglecting the $-$ive sign because $90°-\delta$ is the acute angle between the line OP and the normal to the plane OMN

or $$\sin\delta=\frac{abc}{\sqrt{(a^2+b^2+c^2)}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}} \quad \ldots(3)$$

or $$\delta=\sin^{-1}\frac{abc}{\sqrt{(a^2+b^2+c^2)}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}}.$$

This proves the result (ii).

Again let α be the angle between the line OP and the co-ordinate plane $x=0$, so that $90°-\alpha$ is the angle between the line OP and the normal to the plane $x=0$ whose **d.c.'s are** $1, 0, 0$. Hence we have

$$\cos(90°-\alpha)=\frac{a.1+b.0+c.0}{\sqrt{(a^2+b^2+c^2)}}$$

or $$\sin\alpha=\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \text{ or } \operatorname{cosec}^2\alpha=\frac{a^2+b^2+c^2}{a^2}$$

Similarly, we have

$$\operatorname{cosec}^2\beta=\frac{a^2+b^2+c^2}{b^2} \text{ and } \operatorname{cosec}^2\gamma=\frac{a^2+b^2+c^2}{c^2}$$

$$\therefore \quad \operatorname{cosec}^2\alpha+\operatorname{cosec}^2\beta+\operatorname{cosec}^2\gamma=\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(a^2+b^2+c^2)$$
$$=\frac{(b^2c^2+c^2a^2+a^2b^2)(a^2+b^2+c^2)}{a^2b^2c^2}$$
$$=\operatorname{cosec}^2\delta. \quad \text{[using the relation (3)]}$$

This proves the result (iii).

§ 17. Equations of the planes bisecting the angles between two given planes.

Let the equations of the two given planes be

$$a_1x+b_1y+c_1z+d_1=0, \quad ...($$

and $$a_2x+b_2y+c_2z+d_2=0. \quad ...($$

The equations (1) and (2) should be so written that the con tant terms d_1 and d_2 are both positive. However, we can al write the equations (1) and (2) in such a way that d_1 and d_2 a both negative.

If (x_1, y_1, z_1) be the coordinates of any point on the pla bisecting the angle between the given planes, then the perpend cular distances of this point from both the planes should be equ numerically. Since we consider perpendicular distance as positi when measured in the direction from the origin to the plane, f points on the plane bisecting the angle in which the origin li the perpendicular distances will have the same sign, and for poin on the other bisector, opposite signs. Therefore if the poi (x_1, y_1, z_1) lies on the bisector of the angle *in which the origin li* we have

$$\frac{a_1x_1+b_1y_1+c_1z_1+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}=\frac{a_2x_1+b_2y_1+c_2z_1+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}.$$

Since this relation is satisfied by every point (x_1, y_1, z_1) this bisector, the equation of this bisector plane is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2)^2}}. \quad ...($$

Similarly if (x_1, y_1, z_1) lies on the bisector of the other ang between the two planes, we have

$$\frac{a_1x_1+b_1y_1+c_1z_1+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}=-\frac{a_2x_1+b_2y_1+c_2z_1+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}.$$

The equation of this bisector plane will be

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}} \quad ...($$

Hence the equations (3) and (4) are the required equations the planes bisecting the angles between the given planes. T plane (3) bisects the angle in which the origin lies and the pla (4) bisects the angle in which the origin does not lie. But should not forget to write the equations (1) and (2) in such a w that d_1 and d_2 are of the same sign.

To distinguish between the two bisecting planes as regards t bisector of the acute or obtuse angle between the given planes.

If we are required to find which of the two bisecting pla given by (3) and (4) represents the plane bisecting the acute obtuse angle between the given planes (1) and (2), we find t

value of cos θ, where θ is the acute angle between the bisecting plane and any of the given planes. From this value of cos θ, we find the value of tan θ by using the formula tan $\theta=\sqrt{(\sec^2\theta-1)}$ (or otherwise). In case the value of tan $\theta<1$ then $\theta<\frac{1}{4}\pi$ and hence this bisecting plane bisects the actue angle between the given planes. Again in case the value of tan $\theta>1$ then $\theta>\frac{1}{4}\pi$ and hence this bisecting plane bisects the obtuse angle between the given planes.

An important remark. *To find whether origin lies in the acute or obtuse angle between the glven planes* (1) *and* (2).

The costant terms d_1 and d_2 in the equations (1) and (2) should be of the same sign.

(i) If the **origin** lies in the **acute angle** between the given planes (1) and (2), then the angle θ (say) between the normals to these planes is obtuse and therefore the value of cos θ is negative *i.e.* $\mathbf{a_1a_2+b_1b_2+c_1c_2=}$**negative** is the condition for the origin to lie in the acute angle between the planes.

(ii) If the **origin** lies in the **obtuse angle** between the given planes (1) and (2), then the angle θ (say) between the normals to these planes is acute and therefore the value of cos θ is positive *i.e.,* $\mathbf{a_1a_2+b_1b_2+c_1c_2=}$ **positive** is the condition for the origin to lie in the obtuse angle between the planes.

Thus to distinguish between the two bisecting planes, we should first find whether the origin lies in the acute or obtuse angle by the above method. Then we should find the plane bisecting the angle which contains the origin and the plane bisecting the angle which does not contain the origin. In this way we can distinguish between the two bisecting planes with respect to the position of origin and the acute and obtuse angles.

Solved Examples (3) D

Ex. 1. *Find the equations of the bisectors of the angles between the planes* $2x-y-2z-6=0$ *and* $3x+2y-6z-12=0$ *and distinguish them.*

Sol. Writing the given equations in such a way that the constant terms are both positive, the equations of the given planes are

$$-2x+y+2z+6=0, \quad ...(1)$$

and

$$-3x-2y+6z+12=0. \quad ...(2)$$

The equation of the bisector plane of the angle between the planes (1) and (2) which contains the origin is given by

$$\frac{-2x+y+2z+6}{\sqrt{(4+1+4)}}=\frac{-3x-2y+6z+12}{\sqrt{(9+4+36)}}$$

or $\quad 7(-2x+y+2z+6)=3(-3x-2y+6z+12)$

or $\quad 5x-13y+4z-6=0.$ ...(

The equation of the other bisector plane is

$$\frac{-2x+y+2z+6}{\sqrt{(4+1+4)}}=-\frac{-3x-2y+6z+12}{\sqrt{(9+4+36)}}$$

or $\quad 7(-2x+y+2z+6)=-3(-3x-2y+6z+12)$

or $\quad 23x-y-32z-78=0$ ...(

Now let θ be the acute angle between the plane (4) and th bisecting plane (3). Then

$$\cos\theta=\left|\frac{-2.5+1(-13)+2.4}{\sqrt{(4+1+4)}\sqrt{(25+169+16)}}\right|=\frac{15}{3\sqrt{210}}=\sqrt{\frac{5}{42}}.$$

$\therefore\ \tan\theta=\sqrt{(\sec^2\theta-1)}=\sqrt{[(42/5)-1]}=\sqrt{(37/5)}>1$, so that $\theta>$

Hence the plane $5x-13y+4z-6=0$ bisects the obtuse ang between the given planes (1) and (2) so that the other pla $23x-y-32z-78=0$ bisects the acute angle.

We also note from the given planes (1) and (2), that

$$a_1a_2+b_1b_2+c_1c_2=(-2)(-3)+1(-2)+2(6)$$
$$=6-2+12=16=\text{positive.}$$

Hence the origin lies in the obtuse angle between the gi planes. [See remark to § 1

This confirms that the plane (3) is the bisector of the obtu angle. Hence the equation (4) is the bisector of the acute angle.

Ex. 2. *Show that the origin lies in the acute angle between t planes* $x+2y+2z-9=0$ *and* $4x-3y+12z+13=0$. *Find the pla bisecting the angles between them and point out the one which bise the acute angle.*

Sol. In order that the constant terms are positive, t equations of the given planes may be written as

$$-x-2y-2z+9=0 \quad ...($$

and $\quad 4x-3y+12z+13=0.$

We have

$$a_1a_2+b_1b_2+c_1c_2=(-1).4+(-2).(-3)+(-2).(12)$$
$$=-4+6-24=-22=\text{negative.}$$

Hence the origin lies in the acute angle between the plan (1) and (2) [See remark to § 17].

The equation of the plane bisecting the angle between t given planes (1) and (2) which contains the origin is

$$\frac{-x-2y-2z+9}{\sqrt{(1+4+4)}}=\frac{4x-3y+12z+13}{\sqrt{(16+9+144)}}$$

or $\quad 13(-x-2y-2z+9)=3(4x-3y+12z+13)$

or $\quad 25x+17y+62z-78=0.$ ...(3)

We have proved above that origin lies in the acute angle between the planes and so the equation (3) is the equation of the bisector plane which bisects the acute angle between the given planes.

The equation of the other bisector plane (*i.e.*, the plane bisecting the obtuse angle) is

$$\frac{-x-2y-2z+9}{\sqrt{(1+4+4)}}=-\frac{4x-3y+12z+13}{\sqrt{(16+9+144)}}$$

or $\quad x+35y-10z-156=0.$ ...(4)

$\therefore$ the equations (3) and (4) give the planes bisecting the angles between the given planes and the equation (3) is the bisector of the acute angle.

Ex. 3 *Find the bisector of the acute angle between the planes* $2x-y+2z+3=0$ *and* $3x-2y+6z+8=0$.

Sol. Proced as above in Ex. 1 or Ex 2. The required bisector plane is $\quad 23x-13y+32z+45=0.$

Ex. 4. *Find the equation of the plane that bisects the angle between the planes* $3x-6y+2z+5=0$ *and* $4x-12y+3z=3$ *which contains the origin. Is this the plane that bisects the obtuse angle ?*

Sol. Proceed as above in Ex. 1 or Ex. 2. The required bisecting plane is $67x-162y+47z+44=0$, and this is the bisector of the acute angle.

§ 18. Combined equation of a Pair of planes.

To find the condition that the general homogeneous equation of second degree in x, y and z namely

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

may reprsent a pair of planes and to find the angle between them Also to find the condition of perpendicularity of these planes.

The general homogeneous equation of second degree is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(1)$$

Let the equations of the two planes represented by (1) be

$$l_1x+m_1y+n_1z=0 \text{ and } l_2x+m_2y+n_2z=0.$$

These equations will not contain the constant terms for otherwise their product will not be homogeneous. Thus we have

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy \equiv (l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z).$$

Comparing the coefficients of like terms on either side, we have

$$\left.\begin{array}{l} l_1l_2=a,\ m_1m_2=b,\ n_1n_2=c,\ m_1n_2+m_2n_1=2f, \\ n_1l_2+n_2l_1=2g,\ l_1m_2+l_2m_1=2h \end{array}\right\}. \qquad \ldots(2)$$

The required condition is obtained by eliminating l_1, m_1, n_1 and l_2, m_2, n_2 from the relations (2). This is conveniently done by considering the following product of two zero-valued determinants :

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0.$$

[Remember]

Multiplying the two determinants by row-by-row multiplication rule, we have

$$\begin{vmatrix} 2l_1l_2 & l_1m_2+l_2m_1 & l_1n_2+l_2n_1 \\ m_1l_2+m_2l_1 & 2m_1m_2 & m_1n_2+m_2n_1 \\ n_1l_2+n_2l_1 & n_1m_2+n_2m_1 & 2n_1n_2 \end{vmatrix} = 0.$$

On putting the values of l_1l_2, $l_1m_2+l_2m_1$ etc. from (2), we have

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or $$abc+2fgh-af^2-bg^2-ch^2=0. \qquad \ldots(3)$$

This is the required condition that the equation (1) represents a pair of planes passing through the origin.

To find the angle between the planes. Let θ be the angle between the two planes represented by the equation (1).

Then θ is the angle between the planes $l_1x+m_1y+n_1z=0$ and $l_2x+m_2y+n_2z=0$ and so is given by

$$\tan\theta = \frac{[\Sigma (m_1n_2-m_2n_1)^2]^{1/2}}{l_1l_2+m_1m_2+n_1n_2},$$

where $l_1l_2+m_1m_2+n_1n_2=a+b+c$

and $\Sigma (m_1n_2-m_2n_1)^2=\Sigma [(m_1n_2+m_2n_1)^2-4m_1m_2n_1n_2]$

$$=\Sigma (4f^2-4bc)$$
$$=4(f^2-bc)+4(g^2-ca)+4(h^2-ab),$$

so that $[\Sigma(m_1n_2-m_2n_1)^2]^{1/2}=2\sqrt{(f^2+g^2+h^2-bc-ca-ab)}$.

$$\therefore \quad \tan\theta=\frac{\mathbf{2\sqrt{(f^2+g^2+h^2-bc-ca-ab)}}}{\mathbf{a+b+c}} \qquad \ldots(4)$$

or $$\theta=\tan^{-1}\left[\frac{2\sqrt{(f^2+g^2+h^2-bc-ca-ab)}}{a+b+c}\right]$$

Condition of perpendicularity. The two planes given by (1) will be perpendicular if $\theta=\frac{1}{2}\pi$ *i.e.* $\tan\theta=\tan\frac{1}{2}\pi=\infty$. Hence the relation (4) gives $\mathbf{a+b+c=0}$.

Thus two planes given by (1) will be **perpendicular if the coefficient of x^2 + the coefficient of y^2 + the coefficient of $z^2=0$.**

Solved Examples 3 (E)

Ex. 1. *Prove that the equation $x^2+4y^2-z^2+4xy=0$ represents a pair of planes and find the angle between them.*

Sol. Comparing the given equation with the homogeneous equation of second degree in x, y, z [See equation (1). § 18], we have $a=1, b=4, c=-1, f=0, g=0, h=2$:

$$\therefore \quad abc+2fgh-af^2-bg^2-ch^2$$
$$=1.4.(-1)+2.0.0.2-0-0-(-1)(2)^2=-4+4=0.$$

Hence the given equation represents a pair of planes.

If θ is the angle between the planes, then

$$\tan\theta=\frac{2\sqrt{(f^2+g^2+h^2-bc-ca-ab)}}{a+b+c} \qquad \text{[See result (4), § 18]}$$

$$=\frac{2\sqrt{(0+0+4+4+1-4)}}{1+4-1}, \text{ putting for } a, b, c \text{ etc.}$$

$$=\tfrac{1}{2}\sqrt{5}. \qquad \therefore\ \theta=\tan^{-1}(\tfrac{1}{2}\sqrt{5}).$$

Alternative method. The given equation may be written as

$$(x^2+4xy+4y^2)-z^2=0, \text{ or } (x+2y)^2-z^2=0$$

or $$(x+2y+z)(x+2y-z)=0.$$

$$\therefore \quad x+2y+z=0,\ x+2y-z=0.$$

These being linear equations in x, y, z represent two planes. If θ is the angle between these planes, then

$$\cos\theta=\frac{1.1+2.2+1.(-1)}{\sqrt{(1+4+1)}\sqrt{(1+4+1)}}=\frac{4}{6}=\frac{2}{3}$$

$$\therefore \quad \tan\theta=\tfrac{1}{2}\sqrt{5}.$$

Ex. 2. *Prove that the equation*

$$2x^2-6y^2-12z^2+18yz+2zx+xy=0$$

represents a pair of planes and find the angle between them.

Sol. Comparing the given equation with the equation (1), § 18, we get

$a=2, b=-6, c=-12, f=\frac{1}{2}.18=9, g=\frac{1}{2}.2=1, h=\frac{1}{2}.$

$\therefore \quad abc+2fgh-af^2-bg^2-ch^2$

$=2(-6).(-12)+2.9.1.\frac{1}{2}-2.81+6.1+12.\frac{1}{4}$

$=144+9-162+6+3=162-162=0.$

Hence the given equation represents a pair of planes.

If θ be the angle between the planes, then

$$\tan\theta=\frac{2\sqrt{(f^2+g^2+h^2-bc-ca-ab)}}{a+b+c} \quad \text{[See eqn. (4) of § 18]}$$

$$=\frac{2\sqrt{(81+1+\frac{1}{4}-72+24+12)}}{2-6-12}=\frac{2\sqrt{[\frac{1}{4}.(185)]}}{16}$$

$$=\frac{\sqrt{(185)}}{16}.$$

$$\therefore \quad \sec^2\theta=1+\tan^2\theta=1+\frac{185}{256}=\frac{441}{256}$$

$$\therefore \quad \sec\theta=\frac{21}{16} \text{ or } \cos\theta=\frac{16}{21} \text{ or } \theta=\cos^{-1}\left(\frac{16}{21}\right)$$

Alternative method. Arranging the given equation as a quadratic in x, we have

$$2x^2+x(2z+y)-(6y^2+12z^2-18yz)=0.$$

$$\therefore \quad x=\frac{(2z+y)\pm\sqrt{[(2z+y)^2+4.2.(6y^2+12z^2-18yz)]}}{2.2}$$

or $\quad 4x=2z \quad y\pm\sqrt{[4z^2+4zy+y^2+48y^2+96z^2-144yz]}$

$=-2z-y\pm\sqrt{(49y^2-140yz+100z^2)}$

$=-2z-y\pm\sqrt{(7y-10z)^2}$

$=-2z-y\pm(7y-10z).$

$\therefore \quad 4x=-2z-y+7y-10z$ and $4x=-2z+y-7y+10z$

or $\quad 4x-6y+12z=0$ and $4x+8y-8z=0$

or $\quad 2x-3y+6z=0$ and $x+2y-2z=0.$

These being linear equations in x, y and z represent the planes. If θ is the angle between these planes then using

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}, \text{ we have}$$

$$\cos\theta=\frac{2.1+(-3)42)+6.(-2)}{\sqrt{(4+9+36)}\sqrt{(1+4+4)}}=\frac{-16}{21},$$

giving the obtuse angle between the planes.

If θ is the acute angle between the planes, then $\cos\theta=16/21$

$\therefore \quad \theta=\cos^{-1}(16/21).$

Ex. 3. *Show that the equation* $\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$ *represents a pair of planes.*

Sol. Multiplying the given equation by $(y-z)(z-x)(x-y)$, we have

$$a(z-x)(x-y)+b(y-z)(x-y)+c(y-z)(z-x)=0$$

or $\quad a(zx-yz-x^2+xy)+b(xy-y^2-zx+yz)+c(yz-xy-z^2+zx)=0$

or $\quad ax^2+by^2+cz^2-(b+c-a)yz-(c+a-b)zx-(a+b-c)xy=0. \quad \text{(1)}$

Comparing the equation (1) with the general homogeneous equation of second degree in x, y, z *i.e.*, the equation

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$, [Refer eqn. (1) of § 18] we have $A=a, B=b, C=c, F=-\frac{1}{2}(b+c-a), G=-\frac{1}{2}(c+a-b), H=-\frac{1}{2}(a+b-c).$

[Note that we have used capital letters *A*, *B*, *B* etc. because small letters *a*, *b*, *c* etc. are used in the question.]

The equation (1) will represent a pair of planes if

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}=0. \quad \text{[Refer § 18]}$$

Putting for *A*, *B*, *C* etc. we have

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}=\begin{vmatrix} a & -\frac{1}{2}(a+b-c) & -\frac{1}{2}(c+a-b) \\ -\frac{1}{2}(a+b-c) & b & -\frac{1}{2}(b+c-a) \\ -\frac{1}{2}(c+a-b) & -\frac{1}{2}(b+c-a) & c \end{vmatrix}$$

$$=\begin{vmatrix} 0 & 0 & 0 \\ -\frac{1}{2}(a+b-c) & b & -\frac{1}{2}(b+c-a) \\ -\frac{1}{2}(c+a-b) & -\frac{1}{2}(b+c-a) & c \end{vmatrix},$$

adding the second and third rows to the first row $=0.$

Hence the given equation represents a pair of planes.

Ex. 4. *If the equation*

$$\phi(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

represents a pair of planes, then show that the products of the distances of the two planes, from (α, β, γ) *is*

$$\frac{\phi(\alpha, \beta, \gamma)}{\sqrt{[\Sigma a^2+4\Sigma f^2-2\Sigma bc]}}.$$

Sol. Let the equation

$$\phi(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(1)$$

represent two planes given by

$$l_1x+m_1y+n_1z=0 \quad ...(2) \quad \text{and} \quad l_2x+m_2y+n_2z=0, \quad ...(3)$$

so that we have

$$\phi(x, y, z) \equiv (l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z), \quad ...(4)$$

where $\phi(x, y, z)$ is given by (1).

Comparing the coefficients of like terms on either side of (4), we have

$$\left.\begin{aligned} &l_1l_2=a,\ m_1m_2=b,\ n_1n_2=c \\ &m_1n_2+m_2n_1=2f,\ n_1l_2+n_2l_1=2g,\ l_1m_2+l_2m_1=2h \end{aligned}\right\}. \quad ...(5)$$

Let p_1 and p_2 be the perpendicular distances of the point (α, β, γ) from the planes (2) and (3) respectively.

Then we have

$$p_1=\frac{l_1\alpha+m_1\beta+n_1\gamma}{\sqrt{(l_1^2+m_1^2+n_1^2)}},\ p_2=\frac{l_2\alpha+m_2\beta+n_2\gamma}{\sqrt{(l_2^2+m_2^2+n_2^2)}}.$$

Multiplying, we get

$$p_1p_2=\frac{(l_1\alpha+m_1\beta+n_1\gamma)(l_2\alpha+m_2\beta+n_2\gamma)}{\sqrt{(l_1^2+m_1^2+n_1^2)}\sqrt{(l_2^2+m_2^2+n_2^2)}}. \quad ...(6)$$

Now substituting α, β, γ for x, y, z respectively in (4), we get

$$(l_1\alpha+m_1\beta+n_1\gamma)(l_2\alpha+m_2\beta+n_2\gamma)=\phi(\alpha, \beta, \gamma). \quad ...(7)$$

Also $(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)$

$$=l_1^2l_2^2+m_1^2m_2^2+n_1^2n_2^2+(l_1^2m_2^2+l_2^2m_1^2)+(m_1^2n_2^2+m_2^2n_1^2)+(n_1^2l_2^2+n_2^2l_1^2)$$

$$=a^2+b^2+c^2+\Sigma\{(l_1m_2+l_2m_1)^2-2l_1l_2m_1m_2\}$$

$$=a^2+b^2+c^2+\Sigma(4h^2-2ab), \quad \text{using the relations (5)}$$

$$=a^2+b^2+c^2+(4h^2-2ab)+(4f^2-2bc)+(4g^2-2ca)$$

$$=(a^2+b^2+c^2)+4(f^2+g^2+h^2)-2(bc+ca+ab)$$

$$=\Sigma a^2+4\Sigma f^2-2\Sigma bc.$$

$$\therefore \sqrt{(l_1^2+m_1^2+n_1^2)}\sqrt{(l_2^2+m_2^2+n_2^2)}=\sqrt{[\Sigma a^2+4\Sigma f^2-2\Sigma bc]}. \quad ...(8)$$

Substituting the values from the relations (7) and (8) in (6), we get

$$p_1p_2=\frac{\phi(\alpha, \beta, \gamma)}{\sqrt{[\Sigma a^2+4\Sigma f^2-2\Sigma bc]}}.$$

§ 19. Projection on a plane.

Recall the definitions of the projection of a point and the projection of the segment of a line on a plane (see § 6, § 7 of chapter 2).

Similarly the projection of an area A on a given plane is defined. Let A be an area enclosed by the curve PQR.... Let P', Q', R', ... be the feet of the perpendiculars drawn from P, Q, R, ... to the given plane. Then the projection of the area A enclosed by the curve PQR ... on the given plane is the area A' enclosed by the curve $P'Q'R'$ .. If θ is the angle between the plane of the area A and the plane of projection, then $A'=A\cos\theta$.

Now we shall discuss two theorems on the projections.

Theorem 1. *Let the projections of an area A on the co-ordinate planes yz, zx and xy be* A_x, A_y *and* A_z *respectively, then*

$$A=A_x^2+A_y^2+A_z^2.$$

Proof. Let the direction cosines of the normal to the plane of area A be l, m, n. Also the normal to the yz-plane is x-axis whose d.c.'s are 1, 0, 0. If α be the angle between the plane of area A and the yz-plane, then α is the angle between the normals to these planes and so

$$\cos\alpha=l.1+m.0+n.0=l.$$

Now the projection A_x of the area A on the yz-plane is given by

$$A_x=A\cos\alpha=Al.$$

Similarly we have $A_y=Am$, $A_z=An$.

Squaring and adding, we have

$$A_x^2+A_y^2+A_z^2=A^2(l^2+m^2+n^2)=A^2.1=A^2.$$

Theorem 2. *The projection of a given plane area A on a given plane ξ is equal to the sum of the projections of* A_x, A_y *and* A_z *on the given plane ξ, where* A_x, A_y *and* A_z *are the projections of the area A on the co-ordinate planes viz. yz, zx and xy-planes respectively.*

Proof. Let l, m, n be the d.c.'s of the normal to the plane A, and let l', m', n' be the d.c.'s of the normal to the plane ξ. Now if θ is the angle between these two planes, then

$$\cos\theta=ll'+mm'+nn'. \quad ...(1)$$

Now let the projection of the area A on the plane ξ be A'; then we have

$$A'=A\cos\theta \text{ or } A'=A(ll'+mm'+nn'). \quad ...(2)$$

Also by definition and in view of theorem 1, we have

$$A_x=Al,\ A_y=Am,\ A_z=An. \quad ...(3)$$

From (2), we have

$A'=(Al)\,l'+(Am)\,m'+(An)\,n'$

$=A_x l'+A_y m'+A_z n'$ [using the relations (3)]

= (the projection of the area A_x on the plane ξ)

+ (the projection of the area A_y on the plane ξ)

+ (the projection of the area A_z on the plane ξ).

Proved.

§ 20. Area of a triangle.

To find the area of a triangle ABC the co-ordinates of whose vertices are $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$ and $C\,(x_3, y_3, z_3)$.

Let l, m, n be the d.c.'s of the normal to the plane of the triangle ABC and let Δ denote the area of this triangle.

Let A_x, B_x and C_x be the projections of the three vertices A, B and C respectively on the yz-plane. Clearly the co-ordinates of these points are given by $A_x\,(0, y_1, z_1)$, $B_x(0, y_2, z_2)$, $C_x\,(0, y_3, z_3)$.

Let Δ_x denote the area of the triangle $A_xB_xC_x$ *i.e.* Δ_x is the area of projection of the area Δ on the yz-plane, so that we have

$$\Delta_x=\Delta.l. \quad \ldots(1)$$

Also by the co-ordinate geometry of two dimensions,

$$\Delta_x=\tfrac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \quad \ldots(2)$$

Similarly if Δ_y and Δ_z are the areas of the projections of the area Δ on zx and xy-planes, then

$$\Delta_y=\Delta.m \quad (3) \quad \text{and} \quad \Delta_z=\Delta.n \quad \ldots(4)$$

where
$$\Delta_y=\tfrac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix},\ \Delta_z=\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Squaring (1), (3) and (4) and adding, we get

$$\Delta_x^2+\Delta_y^2+\Delta_z^2=\Delta^2\,(l^2+m^2+n^2)=\Delta^2.1$$

or
$$\Delta^2=\Delta_x^2+\Delta_y^2+\Delta_z^2. \quad \ldots(5)$$

This gives area Δ of the triangle ABC.

Solved Examples 3 (F)

Ex. 1. *Find the area of the triangle whose vertices are $A(1, 2, 3)$, $B\,(2, -1, 1)$ and $C\,(1, 2, -4)$.* **(Agra 1979)**

Sol. Let Δ_x, Δ_y, Δ_z be the areas of the projections of the

area Δ of triangle ABC on the yz, zx and xy-planes respectively. We have

$$\Delta_x=\tfrac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}=\tfrac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 2 & -4 & 1 \end{vmatrix}=\frac{21}{2},$$

$$\Delta_y=\tfrac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}=\tfrac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & -4 & 1 \end{vmatrix}=\frac{7}{2}$$

(numerically),

and
$$\Delta_z=\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=\tfrac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix}=0.$$

$\therefore$ the required area $\Delta=\sqrt{[\Delta_x^2+\Delta_y^2+\Delta_z^2]}$

$$=\sqrt{\left(\frac{441}{4}+\frac{49}{4}+0\right)}=\tfrac{1}{2}\sqrt{(490)} \text{ square units.}$$

Ex. 2. *A plane makes intercepts $OA=a$, $OB=b$ and $OC=c$ respectively on the co-ordinate axes. Show that the area of the triangle ABC is $\frac{1}{2}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}$.*

Sol. The points A, B and C lie on the axes of x, y and z respectively, so that their co-ordinates are $A(a, 0, 0)$ $B(0, b, 0)$, $C(0, 0, c)$.

Let Δ denote the area of the triangle ABC. The projection of triangle ABC on the yz-plane is the triangle OBC and if Δ_x denotes its area then

$$\Delta_x=\tfrac{1}{2}.OB.OC=\tfrac{1}{2}bc. \quad \ldots(1)$$

The projection of the triangle ABC on the zx-plane is the triangle OCA and its area Δ_y is given by

$$\Delta_y=\tfrac{1}{2}.OC.OA=\tfrac{1}{2}ca.$$

Also the projection of the triangle ABC on the xy-plane is the triangle OAB and its area Δ_z is given by

$$\Delta_z=\tfrac{1}{2}.OA.OB=\tfrac{1}{2}ab. \quad \ldots(3)$$

$\therefore$ the area Δ of the triangle ABC is given by

$$\Delta^2+\Delta_x^2+\Delta_y^2+\Delta_z^2=\tfrac{1}{4}\,(b^2c^2+c^2a^2+a^2b^2)$$

or
$$\Delta=\tfrac{1}{2}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}.$$

Ex. 3 *Find the area of the triangle included between the plane $3x-4y+z=12$ and the co-ordinate planes.*

Sol. The equation of the given plane is

$$3x-4y+z=12 \quad \text{or} \quad x/4-y/3+z/12=1 \qquad \ldots(1)$$

The plane (1) meets the co-ordinate axes in the points $A(4, 0, 0)$, $B(0, -3, 0)$ $C(0, 0, 12)$.

Using the notations of Ex. 2 above, we have

$$\Delta_x=\tfrac{1}{2}.OB.OC=\tfrac{1}{2}.(-3).12=18 \text{ (numerically)},$$

$$\Delta_y=\tfrac{1}{2}.OC.OA=\tfrac{1}{2}.12.4 \quad =24,$$

$$\Delta_z=\tfrac{1}{2}.OA.OB=\tfrac{1}{2}.4.(-3) \quad =6\text{(numerically)}.$$

$\therefore$ area Δ of the triangle $ABC=\sqrt{(\Delta_x^2+\Delta_y^2+\Delta_z^2)}$

$$=\tfrac{1}{2}\sqrt{(18^2+24^2+6^2)}=3\sqrt{(26)} \text{ square units.}$$

Ex. 4. *From a point $P(x', y', z')$ a plane is drawn at right angles to OP to meet the co-ordinate axes at A, B and C. Prove that the area of the triangle ABC is $r^5/(2x'y'z')$, where r is the measure of OP.* **(Kanpur 1983, 84 ; M.U. 1990)**

Sol. The d.r.'s of the line joining O (0, 0, 0) and $P(x', y', z')$ are $x'-0$, $y'-0$, $z'-0$ *i.e.*, x', y', z'.

$\therefore$ the equation of the plane through $P(x', y', z')$ and perpendicular to OP is given by

$$x'(x-x')+y'(y-y')+z'(z-z')=0,$$

or $$xx'+yy'+zz'=x'^2+y'^2+z'^2$$

or $$xx'+yy'+zz'=r^2 \quad [\because r=OP=\sqrt{(x'^2+y'^2+z'^2)}]$$

or $$\frac{x}{r^2/x'}+\frac{y}{r^2/y'}+\frac{z}{r^2/z'}=1.$$

The plane (1) meets the co-ordinate axes in the points $A(r^2/x', 0, 0)$, $B(0, r^2/y', 0)$ and $C(0, 0, r^2/z')$.

Let Δ be the area of the triangle ABC. Also let Δ_x be the area of projection OBC on the yz-plane of the area of triangle ABC. We have

$$\Delta_x=\tfrac{1}{2}OB.OC=\tfrac{1}{2}.\frac{r^2}{y'}.\frac{r^2}{z'}=\tfrac{1}{2}.\frac{r^4}{y'z'}. \qquad \ldots(2)$$

Now d.r.'s of the normal to the plane of $\triangle ABC$ *i.e.*, d.r.'s of the line OP are x', y', z', so that the d.c.'s of this normal are

$$x'/r, \; y'/r, \; z'/r. \qquad [\because r=\sqrt{(x'^2+y'^2+z'^2)}]$$

Also d.c.'s of the normal to the plane of triangle OBC *i.e.*, d.c.'s of x-axis are 1, 0, 0.

If α be the angle between the planes of tringles ABC and OBC, we have

$$\cos\alpha=1.(x'/r)+0.(y'/r)+0.(z'/r)=x'r.$$

Now $$\Delta_x=\Delta\cos\alpha.$$

$$\therefore \quad \frac{r^4}{2y'z'}=\Delta.\frac{x'}{r}, \quad \text{or} \quad \Delta=\tfrac{1}{2}\frac{r^5}{x'y'z'}.$$

Exercises

1. A plane makes intercepts 9, 9/2, $-9/2$, upon the coordinate axes. Find the length of the perpendicular from the origin on it. **(Ans. 3)**

2. Show that the four points (0, -1, 0), (2, 1, -1), (1, 1, 1) and (3, 3, 0) are coplanar and hence show that the equation of the plane passing through these points is $4x-3y+2z=3$. **(Agra 1978)**

3. Show that the four points (0, 4, 3), (-1, -5, -3), (-2, -2, 1) and (1, 1, -1) are coplanar.

4. Find the equation of the plane through the origin and parallel to the plane $3x+9y-7z+5=0$. **(Ans.** $3x+9y-7z=0$**)**

5. Show that the planes $3x+4y-5z=9$ and $2x+6y+6z=7$ are at right angles.

Hint Two planes are at right angles if their normals are at right angles. The d.r.'s of the normals to the two given planes are 3, 4 -5 and 2, 6, 6 respectively and we see that these lines are perpendicular.

6. Find the equation of the plane which contains the line of intersection of the planes $x+y+z-6=0$ and $2x+3y+4z+5=0$ and is perpendicular to the plane $4x+5y-3z-8=0$.

(Ans. $x+7y+13z+96=0$**)**

The Straight Lin

§ 1. General equations of a straight line.

We know that every equation of first degree in x, y and always represents a plane (see chapter 3).

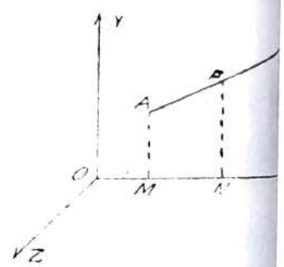
Now let us take two equations of first degree together

i.e. $a_1x+b_1y+c_1z+d_1=0,\ a_2x+b_2y+c_2z+d_2=0.$ ...(

Any point which simultaneously satisfies both the equatio given by (1), will lie on the curve of intersection of the plan given by (1). Since the two planes intersect in a straight lin therefore the equations (1) represent the equations of a straig line. The equations (1) are called *the general equations* of straight line. *Therefore any two equations of first degree in x, and z taken together always represent a straight line.*

§ 2. Symmetrical form of the equations of a straight line.

(A) *To find the equations of a straight line which passes throu a given point (x_1, y_1, z_1) and whose direction cosines are l, m, n.*



Let OX, OY, OZ be chosen as rectangular co-ordinate axes. Let A be the given point (x_1, y_1, z_1) on the line. Choose a general point P with co-ordinates (x, y, z) on the line at a distance r (say) from the given point A. Draw perpendiculars AM and PN on the x-axis from the points A and P respectively, so that MN is the projection of the segment AP on the x-ax and is given by

$$MN=AP\cdot l \quad \text{or} \quad x-x_1=rl.$$

Similarly projecting the segment AP on the y-axis and z-axi

we get $y-y_1=rm$ and $z-z_1=rn$.

Therefore the co-ordinates (x, y, z) of any point P on the line satisfy the equations

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r. \quad \text{...(1)}$$

It should be noted here that 'r' is the **actual distance** of any point $P(x, y, z)$ on the line from the given point (x_1, y_1, z_1).

Hence $$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{l}}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{m}}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{n}} \quad \text{...(2)}$$

are the equations (symmetrical form) of the straight line.

Vector method. We have

$$\overrightarrow{AP}=\text{position vector of } P-\text{position vector of } A$$
$$=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})-(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})$$
$$=(x-x_1)\,\mathbf{i}+(y-y_1)\,\mathbf{j}+(z-z_1)\,\mathbf{k}.$$

Since l, m, n are the direction cosines of the straight line AP, therefore a unit vector along the line AP is $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$.

Now the vector $\overrightarrow{AP}$ is collinear with the unit vector $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$.

$\therefore$ $\overrightarrow{AP}=r\,(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})$, where the scalar r is the distance AP.

$\therefore$ $(x-x_1)\,\mathbf{i}+(y-y_1)\,\mathbf{j}+(z-z_1)\,\mathbf{k}=rl\mathbf{i}+rm\mathbf{j}+rn\mathbf{k}$.

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on both sides, we get

$$x-x_1=rl,\ y-y_1=rm,\ z-z_1=rn.$$

$$\therefore \quad \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$$

are the required equations of the given straight line.

(B) *Symmetrical form in terms of direction ratios.*

Let the direction ratios of the required line be a, b, c. Hence $\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \frac{b}{\sqrt{(a^2+b^2+c^2)}}, \frac{c}{\sqrt{(a^2+b^2+c^2)}}$ are the direction cosines of the line. Thus the equations (2) of the line become

$$\frac{x-x_1}{\dfrac{a}{\sqrt{(a^2+b^2+c^2)}}}=\frac{y-y_1}{\dfrac{b}{\sqrt{(a^2+b^2+c^2)}}}=\frac{z-z_1}{\dfrac{c}{\sqrt{(a^2+b^2+c^2)}}}$$

or $$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{a}}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{b}}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{c}}=\mathbf{r}\ \text{(say)}. \quad \text{...(3)}$$

The equations (3) are the required equations of the straight line. It should be noted here that 'r' is **not the actual distance** of any point $P(x, y, z)$ on the line from the given point (x_1, y_1, z_1).

From equations (2) and (3) we observe that the form of equ tions of the straight line remains unaltered if we use directi ratios instead of direction cosines.

Corollary. From equations (1) and (3), the **general co-ord nates of a point on a line** are given by

(x_1+lr, y_1+mr, z_1+nr) or (x_1+ar, y_1+br, z_1+cr).

(c) The parametric form.

In the equations (1) and (3), r represents a real number whi changes a the position of the point P on the line changes, that r is a parameter. For convenience let this parameter r denoted by 't'. Hence parametric equations of the straight li are given by

$$x=x_1+lt, \; y=y_1+mt, \; z=z_1+nt \quad \ldots($$

or

$$x=x_1+at, \; y=y_1+bt, \; z=z_1+ct \quad \ldots($$

where l, m, n and a, b, c are direction cosines and direction ratio of the line respectively.

Note. If we use the actual direction cosines then the co ordinates of a point (on the line) distant r from the given poi (x_1, y_1, z_1) and (x_1+lr, y_1+mr, z_1+nr). From this point of vie the equations (1) or (2) are also called **distance form** of the equ tions of the straight line.

§ 3. Line through two points.

To find the equations of a straight line passing through tw points whose co-ordinates are (x_1, y_1, z_1) *and* (x_2, y_2, z_2).

Let $A\ (x_1, y_1, z_1)$ and $B\ (x_2, y_2, z_2)$ be the given points throug which the line passes. The direction ratios of the line passin through the points A and B are $x_2-x_1, y_2-y_1, z_2-z_1$.

Hence the required equations of the straight line are give by [from equations (3), § 2 (B)]

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \quad \ldots($$

SOLVED EXAMPLES (A)

Ex 1. *Find the point in which the line* $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ *meets the plane* $x-2y+z=20$.

Sol. The equations of the line are

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=r \text{ (say)}. \quad \ldots(1)$$

The equation of the plane is $x-2y+z=20$. ...(2)

The co-ordinates of any point Q on the line (1) are

$$(2+3r, \; -1+4r, \; 2+12r). \quad \ldots(3)$$

Suppose the line (1) meets the plane (2) in this point Q, hence we have

$$(2+3r)-2(-1+4r)+(2+12r)=20, \text{ or } r=2.$$

Putting this value of r in the co-ordinates of Q given by (3) the co-ordinates of the required point are given by

$$(2+3.2, \; -1+4.2, \; 2+12.2) \text{ or } (8, 7, 26).$$

Ex. 2. *Find the co-ordinates of the point where the line joining the points* $(2, -3, 1)$ *and* $(3, -4, -5)$ *cuts the plane* $2x+y+z=7$.

Sol. The direction ratios of the line joining the points $(2, -3, 1)$ and $(3, -4, -5)$ are $3-2, \; -4-(-3), \; -5-1$ *i.e.* $1, -1, -6$.

Hence the equations of the line joining the given points are

$$\frac{x-2}{3}=\frac{y+3}{-1}=\frac{z-1}{-6}=r, \text{ (say)}. \quad \ldots(1)$$

The co-ordinates of any point on this line are

$$(r+2, \; -r-3, \; -6r+1). \quad \ldots(2)$$

It this point lies on the given plane $2x+y+z=7$. we have

$$2(r+2)+(-r-3)+(-6r+1)=7, \text{ or } r=-1.$$

Putting this value of r in the co-ordinates of the point given by (2), the co-ordinates of the required point are given by $(1, -2, 7)$.

Ex. 3. *Show that the distance of the point of intersection of the line* $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ *and the plane* $x-y+z=5$ *from the point* $(-1, -5, -10)$ *is* 13.

Sol. The equations of the given line are

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}=r \text{ (say)}. \quad \ldots(1)$$

The co-ordinates of any point on the line (1) are $(3r+2, 4r-1, 12r+2)$. If this point lies on the plane $x-y+z=5$, we have $3r+2-(4r-1)+12r+2=5$, or $11r=0$, or $r=0$.

Putting this value of r, the co-ordinates of the point of intersection of the line (1) and the given plane are $(2, -1, 2)$.

$\therefore$ The required distance = distance between the points $(2, -1, 2)$ and $(-1, -5, -10)$

$$=\sqrt{\{(2+1)^2+(-1+5)^2+(2+10)^2\}}$$

$$=\sqrt{(9+16+144)}=\sqrt{(169)}=13.$$

Ex. 4. *Find the points in which the line*

$$\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2} \text{ cuts the surface } 11x^2-5y^2+z^2$$

Sol. The equations of the given line are

$$\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}=r \text{ (say)}.$$

The co-ordinates of any point on the line (1) are

$$(-r-1, 5r+12, 2r+7).$$

If this point lies on the given surface

$$11x^2-5y^2+z^2=0, \text{ we have}$$

$$11(-r-1)^2-5(5r+12)^2+(2r+7)^2=0$$

or $\quad 11(r^2+2r+1)-5(25r^2+120r+144)+(4r^2+28r+49)=0$

or $\quad -110r^2-550r-660=0, \text{ or } r^2+5r+6=0$

or $\quad (r+2)(r+3)=0 \text{ or } r=-2, -3.$

Putting these values of r in (2), the required points of intersection are (1, 2, 3) and (2, −3, 1).

Ex. 5. *Find the equations of the straight lines through the point* (a, b, c) *which are (i) parallel to z-axis (i.e. perpendicular to xy-plane) and (ii) perpendicular to the z-axis (i.e. parallel to xy-plane).*

Sol. Let the equations of any line through the point (a, b, c) be $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$

where l, m, n are the d.c.'s of the line.

(i) The d.c.'s of the z-axis (*i.e.* of the line perpendicular to the xy-plane are 0, 0, 1. Hence if the line (1) is parallel to z-axis or perpendicular to the xy-plane then l, m, n are proportional to 0, 0, 1. Hence the equations of the required line are

$$\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}.$$

(ii) If the line (1) is perpendicular to the z-axis or parallel to the xy-plane then we have

$$l.0+m.0+n.1=0 \text{ or } n=0.$$

Therefore in this case the equations (1) of the required line are given by

$$\frac{x-a}{x}=\frac{y-b}{m}=\frac{z-c}{0}.$$

Ex. 6. *Find the distance of the point (1, −2, 3) from the plane* $x-y+z=5$ *measured parallel to the line* $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$.

Sol. Note here that we are not required to find the perpendicular distance of the point (1, −2, 3) from the given plane, but we are required to evaluate the distance of the point (1, −2, 3) from the given plane measured parallel to a line whose direction cosines are proportional to 2, 3, −6. For this we proceed as follows :

The equations of the line through the point (1, −2, 3) and parallel to the line whose direction cosines are proportional to 2, 3, −6 are given by

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=r \text{ (say)}.$$

The co-ordinates of any point on it are $(2r+1, 3r-2, -6r+3)$.

If this point lies on the given plane $x-y+z=5$, we have

$2r+1-(3r-2)+(-6r+3)=5$, or $-7r=-1$, or $r=1/7$.

$\therefore$ The point of intersection is $\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$

$\therefore$ The required distance = The distance between the points (1, −2, 3) and (9/7, −11/7, 15/7)

$$=\sqrt{\left\{\left(1-\frac{9}{7}\right)^2+\left(-2+\frac{11}{7}\right)^2+\left(3-\frac{15}{7}\right)^2\right\}}$$

$$=1/7\sqrt{\{4+9+36\}}=1/7.7=1.$$

Ex. 6. *Find the equations of the straight lines which bisect the angles between the lines* $x/l_1=y/m_1=z/n_1$, $x/l_2=y/m_2=z/n_2$.

Sol. Proceeding as in Ex. 17 page 38 chapter 2, the direction cosines of the bisectors are proportional to $l_1 \pm l_2$, $m_1 \pm m_2$, $n_1 \pm n_2$. Clearly the given lines pass through the point (0, 0, 0) and hence their bisectors also pass through the point (0, 0, 0) and so the required equations of the bisectors are

$$\frac{x}{l_1 \pm l_2}=\frac{y}{m_1 \pm m_2}=\frac{z}{n_1 \pm n_2}.$$

Ex. 8. *Find the equations of the line through the point* (x_1, y_1, z_1) *at right angles to the lines*

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1} \text{ and } \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}.$$

Sol. Let the equations of the required line through the point (x_1, y_1, z_1) be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}. \qquad \ldots(1)$$

where the d.r.'s l, m, n of this line are to be determined. Since line (1) is perpendicular to the given lines, hence we have

$ll_1+mm_1+nn_1=0$ ...(2) and $ll_2+mm_2+nn_2=0$.

Solving the equations (9) and (3), we have

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}.$$

Putting these proportionate values of l, m, n in (1), the equations of the required line are given by

$$\frac{x-x_1}{m_1n_2-m_2n_1}=\frac{y-y_1}{n_1l_2-n_2l_1}=\frac{z-z_1}{l_1m_2-l_2m_1}.$$

Ex. 9. *Find the equation of the plane through the point (α, β, γ) and (i) perpendicular to the straight line*

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n,$$

(ii) parallel to the lines $x/l_1=y/m_1=z/n_1$ and $x/l_2=y/m_2=z/n_2$.

Sol. Let the equation of any plane through the point (α, β, γ) be $A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0$. ...(1)

(i) The equations of the given line are

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n. \quad ...(2)$$

If the plane (1) is perpendicular to the line (2), then the normal to the plane (1) is parallel to the line (2); and hence A, B, C, the d.r.'s of the normal to the plane (1), are proportional to l, m, n. So the equation of the required plane is given by

$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0.$$

(ii) The equations of the two lines are given as

$$x/l_1=y/m_1=z/n_1 \text{ and } x/l_2=y/m_2=z/n_2.$$

The plane (1) will be parallel to these lines, if its normal whose d.r.'s are A, B, C is perpendicular to both the given lines. Hence we have $Al_1+Bm_1+Cn_1=0$ and $Al_2+Bm_2+Cn_2=0$.

Solving these equations, we have

$$\frac{A}{m_1n_2-m_2n_1}=\frac{B}{n_1l_2-n_2l_1}=\frac{B}{l_1m_2-l_2m_1}$$

Putting these proportionate values of A, B, C in (1), the equation of the required plane is given by

$$(m_1n_2-m_2n_1)(x-\alpha)+(n_1l_2-n_2l_1)(y-\beta)+(l_1m_2-l_2m_1)(z-\gamma)=0.$$

Some Examples on the foot of perpendicular from a point to a plane.

Ex. 10. *Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $3x+4y-6z+1=0$. Find also the co-ordinates of the point on the line which is at the same distance from the foot of the perpendicular as the origin is.*

Sol. The equation of the plane is $3x+4y-6z+1=0$. ...(1)

The direction ratios of the normal to the plane (1) are $3, 4, -6$. Hence the line normal to the plane (1) has d.r.'s $3, 4, -6$, so that the equations of the line through $(0, 0, 0)$ and perpendicular to the plane (1) are

$$x/3=y/4=z/-6=r \text{ (say)}. \quad ...(2)$$

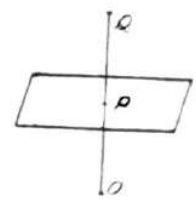
The co-ordinates of any point P on (2) are

$$(3r, 4r, -6r). \quad ...(3)$$

If this point lies on the plane (1), then

$$3(3r)+4(4r)-6(-6r)+1=0, \text{ or } r=-1/61.$$

Putting the value of r in (3), the co-ordinates of the foot of the perpendicular P are $(-3/61, -4/61, 6/61)$.



Now let Q be the point on the line which is at the same distance from the foot of the perpendicular as the origin. Let (x_1, y_1, z_1) be the co-ordinates of the point Q. Clearly P is the middle point of OQ. Hence we have

$$\frac{x_1+0}{2}=-\frac{3}{61}, \frac{y_1+0}{2}=-\frac{4}{61}, \frac{z_1+0}{2}=\frac{6}{61}$$

or $x_1=-6/61, y_1=-8/61, z_1=12/61$.

$\therefore$ The co-ordinates of Q are $(-6/61, -8/61, 12/61)$.

Ex. 11. *Show that if the axes are rectangular, the equations to the perpendicular from the point (α, β, γ) to the plane*

$$ax+by+cz+d=0 \text{ are } (x-\alpha)/a=(y-\beta)/b=(z-\gamma)/c.$$

Deduce the perpendicular distance of the point (α, β, γ) from the plane. Find also the co-ordinates of the foot of the perpendicular.

Sol. The equation of the plane is

$$ax+by+cz+d=0. \quad ...(1)$$

The direction ratios of the normal to the plane (1) are a, b, c. If the line is perpendicular to the plane (1), then it is parallel to the normal to the plane. Hence the d.r.'s of the line perpendicular to the plane (1) are a, b, c. Also it passes through the point (α, β, γ), therefore the required equations of the perpendicular are

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}=r \text{ (say)}. \quad ...(2)$$

The co-ordinates of any point on the line (2) are

$$(br+\alpha, br+\beta, cr+\gamma). \quad ...(3)$$

If this point lies on the plane (1), we have

$$a(ar+\alpha)+b(br+\beta)+c(cr+\gamma)+d=0$$

or $$r(a^2+b^2+c^2)=-(a\alpha+b\beta+c\gamma+d)$$

or $$r=-(a\alpha+a\beta+c\gamma+d)/(a^2+b^2+c^2). \quad ...(4)$$

Putting this value of r from (4) in (3), we get the co-ordinates of the foot of the perpendicular.

Now the perpendicular distance of the point (α, β, γ) from the plane (1) = the distance between the points (α, β, γ) and $(ar+\alpha, br+\beta, cr+\gamma)$

$$=\sqrt{[(ar+\alpha-\alpha)^2+(br+\beta-\beta)^2+(cr+\gamma-\gamma)^2]}$$

$$=r\sqrt{(a^2+b^2+c^2)}$$

$$=\frac{(a\alpha+b\beta+c\gamma+d)}{(a^2+b^2+c^2)}\sqrt{(a^2+b^2+c^2)}=\frac{a\alpha+b\beta+c\gamma+d}{\sqrt{(a^2+b^2+c^2)}}.$$

Ex 12. *Find the equations of the line through* $(1, -1, 2)$ *perpendicular to the plane* $3x+5y-4z=5$ *and deduce the length of the perpendicular from* $(1, -1, 2)$ *upon the plane and also the co-ordinates of the foot of the perpendicular.*

Sol. Proceeding as in Ex. 11 above, we get

(i) the equations of the required line as

$$\frac{x-1}{3}=\frac{y+1}{5}=\frac{z-2}{-4}=r \text{ (say)} \quad ...(1)$$

(ii) The co-ordinates of any point on the line (1) are

$$(3r+1, 5r-1, -4r+2). \quad ...(2)$$

If this point lies on the plane $3x+5y-4z=5$, we have

$$5(3r+1)+5(5r-1)-4(-4r+2)=5, \text{ or } r=3/10.$$

Putting the value of r in (2), the co-ordinates of the foot of the perpendicular are $(19/10, 1/2, 4/5)$.

(iii) The required distance

$$=\sqrt{\left[\left(\frac{19}{10}-1\right)^2+\left(\frac{1}{2}+1\right)^2+\left(\frac{4}{5}-2\right)^2\right]}$$

$$=\sqrt{[(9/10)^2+(3/2)^2+(6/5)^2]}=3\sqrt{}/2.$$

Ex. 13. *Find the incentre of the tetrahedron formed by the planes* $x=0, y=0, z=0$ *and* $x+y+z=a$.

Sol. The three planes $x=0, y=0$ and $z=0$ meet in the point $(0, 0, 0)$. Hence the incentre of the tetrahedron lies on the perpendicular from $(0, 0, 0)$ to the plane $x+y+z=a$.

The d.r.'s of the normal to the plane $x+y+z=a$ are 1, 1, 1. Hence the equations of the perpendicular from $(0, 0, 0)$ to the plane $x+y+z=a$ are

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}=r \text{ (say)}. \quad ...(1)$$

The co-ordinates of any point on (1) are (r, r, r).

If this point lies on the plane $x+y+z=a$, we have

$$r+r+r=a, \text{ or } r=\tfrac{1}{3}a.$$

Hence the foot of the perpendicular is [or the perpendicular from $(0, 0, 0)$ meets the plane $x+y+z=a$ in] $(\frac{1}{3}a, \frac{1}{3}a, \frac{1}{3}a)$.

Let the incentre of the tetrahedron be (x_1, y_1, z_1). Now we know that the incentre divides the join of $(0, 0, 0)$ and $(\frac{1}{3}a, \frac{1}{3}a, \frac{1}{3}a)$ in the ratio 3 : 1 [3 on vertex side and 1 on plane side]. Hence we have

$$x_1=\frac{3.\frac{1}{3}a+1.0}{3+1}=y_1=z_1 \text{ or } x_1=y_1=z_1=\frac{a}{4}$$

∴ The co-ordinates of required incentre are $(\frac{1}{4}a, \frac{1}{4}a, \frac{1}{4}a)$.

Ex. 14. *A variable plane makes intercepts on the co-ordinate axes the sum of whose squares is constant and equal to* k^2. *Show that the locus of the foot of the perpendicular from the origin to the plane is* $(x^{-2}+y^{-2}+z^{-2})(x^2+y^2+z^2)=k^2$.

Sol. Let the equation of the variable plane be

$$x/a+y/b+z/c=1 \quad ...(1)$$

where a, b, c are its intercepts on the co-ordinate axes, so that

$$a^2+b^2+c^2=k^2. \quad ...(2)$$

The d.r.'s of the normal to the plane (1) are $1/a, 1/b, 1/c$. Hence the equations of the line perpendicular to the plane (1) [*i.e.* parallel to the normal to the plane] and passing through the origin are

$$\frac{x-0}{1/a}=\frac{y-0}{1/b}=\frac{z-0}{1/c}=r \text{ (say)}. \quad ...(3)$$

Any point on line (3) is $(r/a, r/b, r/c)$. ...(4)

If this point lies on the plane (1), we have

$$r/a^2+r/b^2+r/c^2=1 \text{ or } r=\frac{1}{a^{-2}+b^{-2}+c^{-2}}.$$

Let (x, y, z) be the co-ordinates of the foot of the perpendicular, then putting the value of r in (4), we get

$$x=\frac{1}{a(a^{-2}+b^{-2}+c^{-2})} \text{ or } x=\frac{a^{-1}}{a^{-2}+b^{-2}+c^{-2}}$$

Similarly $$y=\frac{b^{-1}}{a^{-2}+b^{-2}+c^{-2}} \text{ and } z=\frac{c^{-1}}{a^{-2}+b^{-2}+c^{-2}}.$$

Now $$x^2+y^2+z^2=\frac{a^{-2}+b^{-2}+c^{-2}}{(a^{-2}+b^{-2}+c^{-2})^2}=\frac{1}{a^{-2}+b^{-2}+c^{-2}} \quad ...(5)$$

and $$x^{-2}+y^{-2}+z^{-2}=(a^{-2}+b^{-2}+c^{-2})^2(a^2+b^2+c^2)$$
$$=(a^{-2}+b^{-2}+c^{-2})^2.k^2, \text{ using (2)}. \quad ...(6)$$

Multiplying (5) and (6), we have

$$(x^2+y^2+z^2)^2 (x^{-2}+y^{-2}+z^{-2})=k^2.$$

This is the equation of the required locus.

Ex. 15. *Find the equations of the line through the points (a, b, c) and (a′, b′, c′) and prove that it passes through the origin if $aa'+bb'+cc'=rr'$, where r and r′ are the distances of the point from the origin.*

Sol. The equations of the line through the points (a, b, c) and (a', b', c') are

$$\frac{x-a}{a'-a}=\frac{y-a}{b'-b}=\frac{z-c}{c'-c}. \qquad \ldots(1)$$

The line (1) passes through the origin, hence we have

$$\frac{0-a}{a'-a}=\frac{0-b}{b'-b}=\frac{0-c}{c'-c} \text{ or } \frac{a'-a}{-a}=\frac{b'-b}{-b}=\frac{c'-c}{-c}$$

or $$-\frac{a'}{a}+1=-\frac{b'}{b}+1=-\frac{c'}{c}+1 \text{ or } \frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}.$$

From these relations, we immediately get

$$ab'-a'b=0,\ bc'-b'c=0,\ ca'-c'a=0. \qquad \ldots(2)$$

By Lagrange's indentity, we have

$$(a^2+b^2+c^2)(a'^2+b'^2+c'^2)-(aa'+bb'+cc')^2$$
$$=(ab'-a'b)^2+(bc'-b'c)^2+(ca'-c'a)^2$$
$$=0+0+0$$

or $$(a^2+b^2+c^2)(a'^2+b'^2+c'^2)=(aa'+bb'+cc')^2 \qquad \ldots(3)$$

Now r = the distance of (a, b, c) from the origin

$$=\sqrt{[(a-0)^2+(b-0)^2+(c-0)^2]}=\sqrt{(a^2+b^2+c^2)}.$$

Similarly $r'=\sqrt{(a'^2+b'^2+c'^2)}$.

Putting these values in (3), we have

$$(aa'+bb'+cc')^2=(rr')^2 \text{ or } aa'+bb'+cc'=rr'.$$

This is the required result.

Ex. 16. *Find the image of the point* (1, 3, 4) *in the plane* $2x-y+z+3=0$.

Sol. The equation of the plane is

$$2x-y+z+3=0. \qquad \ldots(1)$$

Let (1, 3, 4) be the point P. Draw a perpendicular PN from the point P to the plane (1). Take a point Q on this perpendicular on the other side of the plane (1). Then Q is called the image of P if N, the foot of the perpendicular, is the middle point of PQ. Let the co-ordinates of Q be (x_1, y_1, z_1).

The d.r.'s of the normal to the plane (1) are 2, −1, 1. Hence the equations of the line through (1, 3, 4) and perpendicular to the plane (1) are

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r \text{ (say)}. \qquad \ldots(2)$$

Any point on (2) is $(2r+1, -r+3, r+4)$. If this point is the foot of the perpendicular *i.e.* the point N, then it will lie on the plane (1) and we have

$$2(2r+1)-(-r+3)+(r+4)+3=0,$$

or $$6r+6=0, \text{ or } r=-1.$$

Putting this value of r, the co-ordinates of N are $(-1, 4, 3)$.

Now as explained above, N is the middle point of PQ. Hence we have

$$-1=\frac{1+x_1}{2},\ 4=\frac{3+y_1}{2},\ 3=\frac{4+z_1}{2}$$

or $$x_1=-3,\ y_1=5,\ z_1=2.$$

∴ The image of P (1, 3, 4) is the point Q (−3, 5, 2).

§ 4. To transform the general form of the equations of a straight line to symmetrical form.

Let the general form of the equations of the straight line be given by the equations

$$\left.\begin{array}{l} a_1x+b_1y+c_1z+d_1=0 \\ a_2x+b_2y+c_2z+d_2=0 \end{array}\right\}. \qquad \ldots(1)$$

Now we are required to write down the symmetrical form of the straight line given by equations (1). For this we must know (i) the direction cosines or direction ratios of the line and (ii) the co-ordinates of a point on the line. To find these two we proceed as follows :

Step 1. *To find direction cosines or the direction ratios of the line given by equations* (1). Let l, m, n be the direction cosines or direction ratios of the line. Since the line is common to both the planes and therefore it is perpendicular to the normals of both the planes. The direction ratios of the normals to the planes given by equations (1) are a_1, b_1, c_1 and a_2, b_2, c_2 respectively. Hence we have

$$la_1+mb_1+nc_1=0 \text{ and } la_2+mb_2+nc_2=0.$$

Solving these equations for l, m, n, we have

$$\frac{l}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}.$$

$\therefore$ the direction ratios of the line are

$$b_1c_2-b_2c_1,\ c_1a_2-c_2a_1,\ a_1b_2-a_2b_1. \qquad \dots(2)$$

Step 2. *To find the co-ordinates of a point on the line given by equations* (1). The co-ordinates of a point on a line can be chosen in many ways. One of these ways is that we choose the point as the one where the line cuts the xy-plane (*i.e.* $z=0$ plane), provided the line is not parallel to the plane $z=0$ *i.e.* provided

$$a_1b_2-a_2b_1\neq 0.$$

Putting $z=0$ in both the equations given by (1), we get

$$a_1x+b_1y+d_1=0,\ a_2x+b_2y+d_2=0.$$

Solving these equations for x, y, we get

$$\frac{x}{b_1d_2-b_2d_1}=\frac{y}{d_1a_2-d_2a_1}=\frac{1}{a_1b_2-a_2b_1}.$$

Hence the co-ordinates of a point on the line (1), where it cuts the $z=0$ plane are

$$\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1},\ \frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1},\ 0\right). \qquad \dots(3)$$

Hence the equations of the line in symmetrical form are

$$\frac{x-\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)}{b_1c_2-b_2c_1}=\frac{y-\left(\dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}\right)}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1}.$$

Note. If $a_1b_2-a_2b_1=0$, then instead of taking $z=0$ we should take the point where the line cuts $x=0$ plane or $y=0$ plane.

SOLVED EXAMPLES (B)

Ex. 1. *Find in symmetrical form the equations of the line*

$$3x+2y-z-4=0=4x+y-2z+3$$

and find its direction cosines.

Sol. The equations of the given line in general form are

$$3x+2y-z-4=0,\ 4x+y-2z+3=0. \qquad \dots(1)$$

Let l, m, n be the d.c.'s of the line. Since the line is common to both the planes, it is perpendicular to the normals to both the planes.

Hence we have

$$3l+2m-n=0,\ 4l+m-2n=0.$$

Solving these, we get $\dfrac{l}{-4+1}=\dfrac{m}{-4+6}=\dfrac{n}{3-8}$

or $$\frac{l}{-3}=\frac{m}{2}=\frac{n}{-5}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(9+4+25)}}=\frac{1}{\sqrt{(38)}}.$$

$\therefore$ the d.c.'s of the line are $-3/\sqrt{(38)},\ 2/\sqrt{(38)},\ -5/\sqrt{(38)}$.

Now to find the co-ordinates of a point on the line given by

(1), let us find the point where it meets the plane $z=0$. Putting $z=0$ in the equations given by (1), we have

$$3x+2y-4=0,\ 4x+y+3=0.$$

Solving these, we get

$$\frac{x}{6+4}=\frac{y}{-16-9}=\frac{1}{3-8},\ \text{or } x=-2,\ y=5.$$

$\therefore$ The line meets the plane $z=0$ in the point $(-2, 5, 0)$.

Therefore the equations of the given line in symmetrical form are

$$\frac{x+2}{-3}=\frac{y-5}{2}=\frac{z-0}{-5}.$$

Ex. 2. *Find the equations to the line through the point* (1, 2, 3) *parallel to the line* $x-y+2z-5=0;\ 3x+y+z-6=0.$ **[Jodbpur 1967]**

Sol. The equations of the given line in general form are

$$x-y+2z-5=0,\ 3x+y+z-6=0. \qquad \dots(1)$$

Let l, m, n be the d.c.'s of this line. Then we have

$$l-m+2n=0,\ 3l+m+n=0.$$

Solving these, we have

$$\frac{l}{-1-1}=\frac{m}{2.3-1.1}=\frac{n}{1.1+1.3},\ \text{or}\ \frac{l}{-3}=\frac{m}{5}=\frac{n}{4}.$$

Since the required line is parallel to the line (1), the d.c.'s of the required line are proportional to l, m, n *i.e.* $-3, 5, 4$. Hence the equations of required line are given by

$$(x-1)/-3=(y-2)/5=(z-3)/4.$$

Ex. 3. *Find the equations of the line through the point* (x_1, y_1, z_1) *and parallel to the line*

$$a_1x+b_1y+c_1z+d_1=0,\ a_2x+b_2y+c_2z+d_2=0.$$

Sol. Let l, m, n be the d.c.'s of the given line, then proceeding as in § 4, we get

$$\frac{l}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}. \qquad \dots(1)$$

The required line passes through (x_1, y_1, z_1) and its d.c.'s are proportional to l, m, n given by (1). Hence the equations of the required line are given by

$$\frac{x-x_1}{b_1c_2-b_2c_1}=\frac{y-y_1}{c_1a_2-c_2a_1}=\frac{z-z_1}{a_1b_2-a_2a_1}.$$

Ex. 4. *Find in symmetrical form the equations of the line*

$$x=ay+b,\ z=cy+d.$$

Sol. The equations of the given line may be written as

$$x-ay+0.z-b=0,\ 0.x+cy-z+d=0. \quad \ldots(1)$$

Let l, m, n be the d.c.'s of the line. Then we have

$$1.\,l-a.m+0.n=0,\ 0.l+c.m-1.n=0.$$

$$\therefore\ l=an,\ cm=n, \text{ so that } \frac{l}{a}=\frac{m}{1}=\frac{n}{c}. \quad \ldots(2)$$

Now let us find the point where the line (1) meets the plane $y=0$. Putting $y=0$ in the equations (1), we get $x=b$ and $z=d$.

$\therefore$ The line meets the plane $y=0$ in the point $(b, 0, d)$.

Therefore the equations of the given line in symmetrical form are

$$(x-b)/a=(y-0)/1=(z-d)/c.$$

Alternative method. The given equations of the line may be written as

$$x-b=ay,\ z-d=cy$$

or
$$\frac{x-b}{a}=\frac{y}{1},\ \frac{z-d}{c}=\frac{y}{1}.$$

$$\therefore\ \frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}.$$

These are the required equations of the line in symmetrical form.

Ex. 5. *Show that the lines*

$$x=ay+b,\ z=cy+d \text{ and } x=a'y+b',\ z=c'y+d'$$

are perpendicular if $aa'+cc'+1=0$.

Sol. Proceeding as in Ex. 4 above, if the d.c.'s of the line $x=ay+b$, $z=cy+d$ are l_1, m_1, n_1, we have

$$l_1/a'=m_1/1=n_1/c'. \quad \ldots(2)$$

Similarly if l_2, m_2, n_2 are the d.c.'s of the line $x=a'y+b'$, $z=c'y+d'$, then we have

$$l_2/a'=m_2/1=n_2/c.'. \quad \ldots(2)$$

The given lines will be perpendicular, if

$$l_1l_2+m_1m_2+n_1n_2=0$$

i.e. $\quad aa'+1.1+cc'=0$, or $aa'+cc'+1=0$.

§ 5. The plane and the straight line :

To find the co-ordinate of the point of intersection of a given line and a given plane and to deduce the conditions that :

(i) the line may be parallel to the plane,

(ii) the line may be perpendicular to the plane, and

(iii) the line may be lying in the plane

Let the equations of the given line in symmetrical form be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \text{ (say)} \quad \ldots(1)$$

and the equation of the given plane be

$$ax+by+cz+d=0. \quad \ldots(2)$$

The co-ordinates of any point P on the line (1) are

$$(x_1+lr,\ y_1+mr,\ z_1+nr).$$

If this point P lies on the plane (2), we have

$$a(x_1+lr)+b(y_1+mr)+c(z_1+nr)+d=0$$

or $\quad r(al+bm+cn)=-(ax_1+by_1+cz_1+d)$

or $\quad r=-(ax_1+by_1+cz_1+d)/(al+bm+cn).$

Thus the co-ordinates of the p[oint] of intersection [of the] line (1) and the plane (2) are $(x_1+lr,\ y_1+mr,\ z_1+nr)$ whe[re r is] givn by (3).

(i) The conditions of paralelism. If the line (1) is para[llel] to the plane (2), then this line must be perpendicular to the no[r]mal to the plane (2) and hence we have $\quad a+bm+cn=0$.

Again the point (x_1, y, z_1) should not lie on the plane *i.e.* [we] must have $ax_1+by_1+cz_1+d\neq0$, for otherwise the line (1) will n[ot] be simply parallel to the plane (2) but it will lie in the plane (2).

Hence the required conditions that the line (1) is parall[el to] the plane (2) are

$$\mathbf{al+bm+cn=0,\ ax_1+by_1+cz_1+d\neq0.}$$

(ii) The condition of perpendicularity. If the line [(1) is] perpendicular to the plane (2), then the line (1) must be para[llel] to the normal of the plane (2) and hence the required conditi[on] of perpendicularity is given by

$$\frac{\mathbf{a}}{\mathbf{l}}=\frac{\mathbf{b}}{\mathbf{m}}=\frac{\mathbf{c}}{\mathbf{n}}.$$

(iii) The conditions that the line may lie in the plane. If the line (1) lies in the plane (2), then for all values of r the point $P(x_1+lr,\ y_1+mr,\ z_1+nr)$ will lie on the plane (2) *i.e.*

$$a(x_1+lr)+b(y_1+mr)+c(z_1+nr)d=0$$

or $\quad r(al+bm+cn)+(ax_1+by_1+cz_1+d)=0$

should be true for all values of r, and hence we must have

$$al+bm+cn=0 \text{ and } ax_1+by_1+cz_1+d=0.$$

Hence the required conditions that the line (1) lies in the plane (2) are

$$al+bm+cn=0,\ ax_1+by_1+cz_1+d=0$$

These conditions can also be deduced as follows :

If the line (1) lies in the plane (2), then the line is perpendicular to the normal to the plane (2), for which we have

$$al+bm+cn=0.$$

Again the point (x_1, y_1, z_1) must lie on the plane (2) [since the line (1) is passing through the point (x_1, y_1, z_1)], hence we have

$$ax_1+by_1+cz_1+d=0.$$

SOLVED EXAMPLES (C)

Ex. 1. *Find the equation of the plane through the line*

$$P\equiv ax+by+cz+d=0,\ Q\equiv a'x+b'y+c'z+d'=0$$

and parallel to the line $x/l=y/m=z/n$.

Sol. The equation of any plane through the line $P=0$, $Q=0$ *i.e.*, through the line of intersection of the planes $P=0$ and $Q=0$ is

$$P=\lambda Q=0 \qquad \ldots(1)$$

or $\quad (ax+by+cz+d)+\lambda\,(a'x+b'y+c'z+d')=0$

or $\quad (a+\lambda a')\,x+(b+\lambda b')\,y+(c+\lambda c')\,z+d+\lambda d'=0. \qquad \ldots(2)$

The d.c.'s of the normal to the plane (2) are proportional to $a+\lambda a', b+\lambda b', c+\lambda c'$. The plane (2) [or (1)] will be parallel to the line $x/l=y/m=z/n$ if the normal is perpendicular to the line $x/l=y/m=z/n$, hence we have

$$(a+\lambda a')\,l+(b+\lambda b')\,m+(c+\lambda c')\,n=0$$

or $\quad \lambda(a'l+b'm+c'n)=-(al+bm+cn)$

or $\quad \lambda=-(al+bm+cn)/(a'l+b'm+c'n).$

Putting this value of λ in (1), the required equation of the plane is given by

$$P-\{(al+bm+cn)/(a'l+b'm+c'n)\}\,Q=0$$

or $\quad P\,(a'l+b'm+c'n)=Q\,(al+bm+cn).$

Ex. 2. *Find the equation of the plane through the line* $3x-4y+5z=10,\ 2x+2y-3z=4$ *and parallel to the line* $x=2y=3z$.

Sol. The equations of the given line are

$$3x-4y+5z=10,\ 2x+2y-3z=4. \qquad \ldots(1)$$

The equation of any plane through the line (1) is

$$(3x-4y+5z-10)+\lambda\,(2x+2y-3z-4)=0$$

or $\quad (3+2\lambda)\,x+(-4+2\lambda)\,y+(5-3\lambda)\,z-10-4\lambda=0. \qquad \ldots(2)$

The plane (1) will be parallel to the line

$$x=2y=3z \quad i.e. \quad \frac{x}{6}=\frac{y}{3}=\frac{z}{2} \quad \text{if}$$

$$(3+2\lambda).6+(-4+2\lambda).3+(5-3\lambda).2=0$$

or $\quad \lambda\,(12+6-6)+18-12+10=0$, or $\lambda=-\frac{4}{3}$.

Putting this value of λ in (2), the required equation of the plane is given by

$$(3-\tfrac{8}{3})\,x+(-4-\tfrac{8}{3})\,y+(5+4)\,z-10+\tfrac{16}{3}=0,$$

or $\quad x-20y+27z=14.$

Ex. 3. *Find the direction cosines of the line whose equations are* $x+y=3$ *and* $x+y+z=0$ *and show that it makes an angle of* $30°$ *with the plane* $y-z+2=0$.

Sol. The equations of the line are

$$x+y=3,\ x+y+z=0. \qquad \ldots(1)$$

Let l, m, n be the d.c.'s of the line (1), then we have

$$l+m+0.n=0,\ l+m+n=0.$$

Solving, we get

$$\frac{l}{1}=\frac{m}{-1}=\frac{n}{0}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(1)^2+(-1)^2+0^2\}}}=\frac{1}{\sqrt{2}}.$$

$\therefore \quad l=1/\sqrt{2},\ m=-1/\sqrt{2},\ n=0.$

The equation of the plane is $y-z+2=0. \qquad \ldots(2)$

The d.r.'s of the normal to the plane (2) are $0, 1, -1$, *i.e.* direction cosines are $0, 1/\sqrt{2}, -1/\sqrt{2}$.

Now suppose θ is the acute angle between the line (1) and the normal to the plane (2).

Then using the formula $\cos\theta=l_1l_2+m_1m_2+n_1n_2$, we have

$$\cos\theta=\left|\frac{1}{\sqrt{2}}.0+\left(-\frac{1}{\sqrt{2}}\right).\frac{1}{\sqrt{2}}+0.\left(-\frac{1}{\sqrt{2}}\right)\right|=\frac{1}{2}.$$

$\therefore \quad \theta=60°.$

Now the angle between the line (1) and the plane (2) is the complement of the angle θ. So the required angle between the line (1) and the plane (2) is $90°-\theta$ *i.e.*, $90°-60°$ *i.e.*, $30°$.

Ex. 4. *Find the equation of the plane through the points*

$$(2, -1, 0),\ (3, -4, 5)$$

and parallel to the line $3x=2y=z$.

Sol. The equation of any plane through the point $(2, -1, 0)$ is $a\,(x-2)+b\,(y+1)+c\,(z-0)=0. \qquad \ldots(1)$

If the plane (1) passes through the point $(3, -4, 5)$, we get

$$a\,(3-2)+b\,(-4+1)+c\,(5)=0,\ \text{or}\ a-3b+5c=0. \qquad \ldots(2)$$

The equations of the line are $3x=2y=z$, or $\dfrac{x}{2}=\dfrac{y}{3}=\dfrac{z}{6}$. $\qquad \ldots(3)$

The plane (1) will be parallel to the line (3), if

$$2a+3b+6c=0. \qquad \ldots(4)$$

Solving (2) and (4), we get $\frac{a}{-33}=\frac{b}{4}=\frac{c}{9}$.

Putting these proportionate values of a, b, c in (1), the required equation of the plane is given by

$$-33(x-2)+4(y+1)+9(z-0)=0, \text{ or } 33x+4y-9z-70=0$$

Ex. 5. *Find the equation of the plane through (2, 1, 4) perpendicular to the line of intersection of the planes*

$$3x+4y+7z+4=0 \text{ and } x-y+2z+3=0$$

Sol. Let l, m, n be the d.c.'s of the line of intersection of the two planes $3x+4y+7z+4=0$, $x-y+2z+3=0$.

Then we have $3l+4m+7n=0$, $l-m+2n=0$.

Solving, $\frac{l}{8+7}=\frac{m}{7-6}=\frac{n}{-3-4}$, or $\frac{l}{15}=\frac{m}{1}=\frac{n}{-7}$.

Thus d.r.'s of the normal to the required plane are 15, 1, -7. Also the required plane is to pass through the point (2, 1, 4). Hence its equation is

$$15(x-2)+1(y-1)-7(z-4)=0$$

or

$$15x+y-7z-3=0.$$

Ex. 6. *Prove that the lines* $3x+2y+z-5=0=x+y-2z-3$ *and* $2x-y-z=0=7x+10y-8z-15$ *are perpendicular.*

Sol. Let l_1, m_1, n_1 be the d.c.'s of the first line. Then $3l_1+2m_1+n_1=0$, $l_1+m_1-2n_1=0$. Solving, we get

$$\frac{l_1}{-4-1}=\frac{m_1}{1+6}=\frac{n_1}{3-2} \text{ or } \frac{l_1}{-5}=\frac{m_1}{7}=\frac{n_1}{1}.$$

Again let l_2, m_2, n_2 be the d.c.'s of the second line, then

$$2l_2-m_2-n_2=0, \ 7l_2+10m_2-8n_2=0.$$

Solving, $\frac{l_2}{8+10}=\frac{m_2}{-7+16}=\frac{n_2}{20+7}$ or $\frac{l_2}{2}=\frac{m_2}{1}=\frac{n_2}{3}$.

Hence the d.c.'s of the two given lines are proportional to $-5, 7, 1$ and $2, 1, 3$. We have

$$-5.2+7.1+1.3=0$$

$\therefore$ the given lines are perpendicular.

§ 6. *To find the equation of the plane through a given line whose equations are given in (i) general form and (ii) symmetrical form.*

(i) Let the equations of the line in general form be given by

$P \equiv a_1x+b_1y+c_1z+d_1=0$, $Q \equiv a_2x+b_2y+c_2z+d_2=0$: ...(1)

Then $P+\lambda Q=0$ *i.e.*,

$$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

is an equation of first degree in x, y and z and hence represents a

plane. Also $P+\lambda Q=0$ is satisfied by all those points which satisfy (1). Hence the equation of the required plane is given by

$$\mathbf{P+\lambda Q=0.}$$

(ii) Let the equations of the line in symmetrical form be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}. \quad ...(2)$$

The required plane passes through the line (2) and hence it passes through the point (x_1, y_1, z_1) which is a point on the line (2).

The equation of any plane through the point (x_1, y_1, z_1) is given by

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0. \quad ...(3)$$

Again, the required plane passes through the line (2), hence the normal to the plane (3) is perpendicular to the line (2).

$$\therefore \quad al+bm+cn=0.$$

Therefore, the required equation of the plane is

$\mathbf{a(x-x_1)+b(y-y_1)+c(z-z_1)=0}$, **where** $\mathbf{al+bm+cn=0}$.

§ 7. *To find the equation of the plane through a given line and parallel to another line.*

Let the equations of the given line be

$$(x-x_1)/l_1=(y-y_1)/m_1=(z-z_1)/n_1. \quad ...(1)$$

Proceeding as in § 6 (ii), the equation of any plane through the line (1) is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0, \quad ...(2)$$

where $al_1+bm_1+cn_1=0$. ...(3)

Again let the plane (2) be parallel to another given line whose equations are given by

$$(x-x_2)/l_2=(y-y_2)/m_2=(z-z_2)/n_2. \quad ...(4)$$

Since the plane (2) is parallel to the line (4), the normal to the plane (2) will be perpendicular to the line (4) and so we have

$$al_2+bm_2+cn_2=0. \quad ...(5)$$

Eliminating a, b, c between the equations (2), (3), (5), the required equation of the plane is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

or

$$(x-x_1)(m_1n_2-m_2n_1)+(y-y_1)(n_1l_2-n_2l_1)+(z-z_1)(l_1m_2-l_2m_1)=0.$$

SOLVED EXAMPLES (D)

Ex. 1. *Find the equations of the planes through the line $(x-2)/2=(y-3)/3=(z-4)/5$, are parallel to the co-ordinate axes.*

Sol. The equations of the given line are

$$(x-2)/2=(y-3)/3=(z-4)/5 \quad \ldots(1)$$

The equation of any plane through the line (1) [see § 6] is

$$a(x-2)+b(y-3)+c(z-4)=0 \quad \ldots(2)$$

where $2a+3b+5c=0.$ ...(3)

We are required to find the equations of the planes parallel to x-axis, y-axis and z-axis respectively.

(i) The d.c.'s of x-axis are 1, 0, 0. If the plane (2) is parallel to the x-axis, then the normal to the plane (2) is perpendicular to the x-axis *i.e.*, we have

$$a.1+b.0+c.0=0. \quad \ldots(4)$$

Solving (3) and (4), we have

$$\frac{a}{0}=\frac{b}{5-0}=\frac{c}{0-3} \text{ or } \frac{a}{0}=\frac{b}{5}=\frac{c}{-3}.$$

Putting these proportionate values of a, b, c in (2), the required equation of the plane is given by

$$0.(x-2)+5(y-3)-3(z-4)=0,$$

or $5y-3z-3=0.$

(ii) The d.c.'s of the y-axis are 0, 1, 0. If the plane (2) is parallel to the y-axis, we have

$$a.0+b.1+c.0=0. \quad \ldots(5)$$

Solving (3) and (5), we get

$$\frac{a}{0-5}=\frac{b}{0}=\frac{c}{2-0} \text{ or } \frac{a}{-5}=\frac{b}{0}=\frac{c}{2}$$

Putting these proportionate values in (2), the required equation of the plane is given by

$$-5(x-2)+0.(y-3)+2(z-4)=0, \text{ or } 5x-2z-2=0.$$

(iii) The d.c.'s of the z-axis are 0, 0, 1. If the plane (2) is parallel to the z-axis, we have

$$a.0+b.0+c.1=0. \quad \ldots(6)$$

Solving (3) and (6), we get $\frac{a}{3}=\frac{b}{-2}=\frac{c}{0}$.

Putting these proportionate values of a, b, c in (2), the required equation of the plane is given by

$$3(x-2)-2(y-3)=0, \text{ or } 3x-2y=0.$$

Ex. 2. *Prove that the equation of the plane through the line $(x-1)/3=(y+6)/4=(z+1)/2$ and parallel to $(x-2)/2=(y-1)/-3$*

$=(z+4)/5$ is $25x-11y-17z-109=0$ and show that the point $(2, 1, -4)$ lies on it.

Sol. The equations of the given line are

$$(x-1)/3=(y+6)/4=(z+1)/2. \quad \ldots(1)$$

The equation of any plane through the line (1) is

$$a(x-1)+b(y+6)+c(z+1)=0 \quad \ldots(2)$$

where $3a+4b+2c=0.$ ...(3)

The plane (2) is to be parallel to the line

$$(x-2)/2=(y-1)/(-3)=(z+4)/5. \quad \ldots(4)$$

Hence the normal to the plane (2) is perpendicular to the line (4), so that we have

$$2a-3b+5c=0. \quad \ldots(5)$$

Solving (3) and (5), we get

$$\frac{a}{20+6}=\frac{b}{4-15}=\frac{c}{9-8} \text{ or } \frac{a}{26}=\frac{b}{-11}=\frac{c}{-17}$$

Putting these proportionate values of a, b, c in the equation (2), the required equation of the plane is

$$26(x-1)-11(y+6)-17(z+1)=0$$

or $26x-11y-17z-119=0.$

Substituting the point $(2, 1, -4)$ in the equation (6) of the plane, we get $26\times2-11\times1-17\times-4-119=0$, or $0=0$ *i.e.* the point $(2, 1, -4)$ satisfies the equation (6) of the plane.

Remark. In the above Ex. 2, the point $(2, 1, -4)$ lies on the line (4) and also on the plane (6). Hence the line (4) will wholly lie on the plane (6).

Therefore both the lines (1) and (4) are coplanar and the equation (6) gives the plane containing both of them.

Ex. 3. *Find the equation of the plane which contains the two parallel lines*

$$\frac{x+1}{3}=\frac{y-2}{2}=\frac{z}{1} \text{ and } \frac{x-3}{3}=\frac{y+4}{2}=\frac{z-1}{1}.$$

Sol. The equations of the two parallel lines are

$$(x+1)/3=(y-2)/2=(z-0)/1 \quad \ldots(1)$$

and $(x-3)/3=(y+4)/2=(z-1)/1.$ ...(2)

The equation of any plane through the line (1) is

$$a(x+1)+b(y-2)+cz=0, \quad \ldots(3)$$

where $3a+2b+c=0.$ ...(4)

The line (2) will also lie on the plane (3) if the point $(3, -4, 1)$ lying on the line (2) also lies on the plane (3), and for this we have $a(3+1)+b(-4-2)+c.1=0$ or $4a-6b+c=0$. ...(5)

Solving (4) and (5), we get $\frac{a}{8}=\frac{b}{1}=\frac{c}{-26}$.

Putting these proportionate values of a, b, c in (3), the required equation of the plane is

$$8(x+1)+1.(y-2)-26z=0, \text{ or } 8x+y-26z+6=0.$$

Ex. 4. *Find the equation of the plane which contains the line* $x=\frac{1}{2}(y-3)=\frac{1}{3}(z-5)$ *and which is perpendicular to the plane* $2x+7y-3z=1$.

Sol. The equations of the given line are

$$(x-0)/1=(y-3)/2=(z-5)/3. \quad ...(1)$$

The equation of any plane through the line (1) is

$$a(x)+b(y-3)+c(z-5)=0 \quad ...(2)$$

where $1a+2.b+3.c=0$. ...(3)

Also the plane (2) will be perpendicular to the plane

$$2x+7y-3z=1$$

if $2.a+7.b-3.c=0$. ...(4)

Solving (3) and (4), we get

$$\frac{a}{-6-12}=\frac{b}{6+3}=\frac{c}{7-4} \text{ or } \frac{a}{-9}=\frac{b}{3}=\frac{c}{1}.$$

Putting these proportionate values of a, b, c in (2), the required equation of the plane is

$$-9x+3(y-3)+(z-5)=0, \text{ or } 9x-3y-z+14=0.$$

Ex. 5. *Show that the equation to the plane containing the line* $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ *and the point* $(0, 7, -7)$ *is* $x+y+z=0$. *Hence show that the line* $\frac{x}{1}=\frac{y-7}{2}=\frac{z+7}{-3}$ *also lies in the same plane.*

Sol. The equations of the given line are

$$(x+1)/-3=(y-3)/2=(z+2)/1. \quad ...(1)$$

The equation of any plane through the line (1) is

$$a(x+1)+b(y-3)+c(z+2)=0 \quad ...(2)$$

where $-3a+2b+1.c=0$. ...(3)

The plane (2) is also to pass through the point $(0, 7, -7)$.

$\therefore \quad a+4b-5c=0$. ...(4)

Solving (3) and (4), we get

$$\frac{a}{-10-4}=\frac{b}{1-15}=\frac{c}{-12-2} \text{ or } \frac{a}{1}=\frac{b}{1}=\frac{c}{1}.$$

Putting these proportionate values of a, b, c in (2), the required equation of the plane is

$$1.(x+1)+1.(y-3)+1.(z+2)=0 \text{ or } x+y+z=0. \quad ...(5)$$

The equations of the second line are given to be

$$\frac{x}{1}=\frac{y-7}{2}=\frac{z+7}{3}. \quad ...(6)$$

The line (6) passes through the point $(0, 7, -7)$ and this point also lies on the plane (5). Now the line (6) will lie in the plane (5) if the normal to the plane (5) [whose d.r.'s are 1, 1, 1] is perpendicular to the line (6), the condition for which is

'$a_1a_2+b_1b_2+c_1c_2$'$=0$, *i.e.* $1.1+1.2+1.(-3)=0$,

which holds good. Hence the line (6) also lies in the plane (5). This proves the required statement.

Ex. 6. *Find the equation to the plane through the point* $(\alpha', \beta', \gamma')$ *and the line* $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$.

(Gorakhpur 1981 ; Ranchi 68)

Sol. The equation of any plane through the given line is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0 \quad ...(1)$$

where $a.l+b.m+c.n=0$. ...(2)

The plane (1) will pass through the point $(\alpha', \beta', \gamma')$ if

$$a(\alpha'-\alpha)+b(\beta'-\beta)+c(\gamma'-\gamma)=0. \quad ...(3)$$

The equation of the required plane will be obtained by eliminating a, b, c between the equations (1), (3) and (2). Hence eliminating the constants a, b, c between the above equations, the equation of the required plane is given by

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \end{vmatrix}=0$$

or $\Sigma(x-\alpha)\{n(\beta'-\beta)-m(\gamma'-\gamma)\}=0$.

Ex. 7. *Show that the plane through the point* (α, β, γ) *and the line* $x=py+q=rz+s$ *is given by*

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0.$$

(Meerut 1983 S)

Sol. The equations of the given line are

$$x=py+q=rz+s,$$

or $\frac{x-0}{1}=\frac{y+(q/p)}{1/p}=\frac{z+(s/r)}{1/r}$. ...(1)

The equation of any plane through the line (1) is

$$a(x-0)+b(y+q/p)+c(z+s/r)=0 \quad ...(2)$$

where $1.a+(1/p).b+(1/r).c=0$. ...(3)

The plane (2) will also pass through the point (α, β, γ) if

$$a\alpha+b(\beta+q/p)+c(\gamma+s/r)=0.$$

The equation of the required plane is obtained by eliminating the constants a, b, c between the equations (2), (4) and (3) and hence it is given by

$$\begin{vmatrix} x & y+q/p & z+s/r \\ \alpha & \beta+q/p & \gamma+s/r \\ 1 & 1/p & 1/r \end{vmatrix}=0.$$

Multiplying second and third columns by p and r respectively we get the required equation as

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0.$$

Ex. 8. *Show that the equation of any plane through the line* $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ *is* $(x-\alpha)\dfrac{\lambda}{l}+(y-\beta)\dfrac{\mu}{m}+(z-\gamma)\dfrac{\nu}{n}=0$

where $\lambda+\mu+\nu=0.$

Sol. The equations of the line are

$$(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n.$$

The equation of any plane through the line (1) is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$$

where $al+bm+cn=0.$

Now choosing the values $a=\lambda/l,\ b=\mu/m,\ c=\nu/n,$ the equation (2) of the plane is given by

$$(x-\alpha)\,(\lambda/l)+(y-\beta)\,(\mu/m)+(z-\gamma)\,(\nu/n)=0$$

where $\dfrac{\lambda}{l}\cdot l+\dfrac{\mu}{m}\cdot m+\dfrac{\nu}{n}\cdot n=0,$ or $\lambda+\mu+\nu=0.$

This proves the required result.

Ex. 9. *Find the equation of the plane through the line*

$$ax+by+cz=0=a'x+b'y+c'z$$

and $\alpha x+\beta y+\gamma z=0=\alpha'x+\beta'y+\gamma'z.$

Sol. The equations of the first line are

$$ax+by+cz=0=a'x+b'y+c'z.$$

This line clearly passes through the origin. If l, m, n be the d.c.'s of this line then we have

$$al+bm+cn=0 \quad\text{and}\quad a'l+b'm+c'n=0.$$

Solving these,

$$\frac{l}{bc'-b'c}=\frac{m}{ca'-c'a}=\frac{n}{ab'-a'b}.$$

Hence the symmetrical form of the first line is

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}. \quad \ldots(1)$$

The equations of the second line are

$$\alpha x+\beta y+\gamma z=0=\alpha'x+\beta'y+\gamma'z.$$

This line also passes through the origin. Hence its symmetrical form is

$$\frac{x}{\beta\gamma'-\beta'\gamma}=\frac{y}{\gamma\alpha'-\gamma'\alpha}=\frac{z}{\alpha\beta'-\alpha'\beta}. \quad \ldots(2)$$

The equation of any plane through the line (1) is

$$Ax+By+Cz=0 \quad \ldots(3)$$

where $A(bc'-b'c)+B(ca'-c'a)+C(ab'-a'b)=0. \quad \ldots(4)$

Also if the line (2) lies in the plane (3) then the following two conditions must hold :

I. The line (2) passes through the point (0, 0, 0) and so the plane (3) should also pass through (0, 0, 0) and clearly it is true.

II. The normal to the plane (3) should be perpendicular to the line (2), the condition for which is

$$A(\beta\gamma'-\beta'\gamma)+B(\gamma\alpha'-\gamma'\alpha)+C(\alpha\beta'-\alpha'\beta)=0. \quad \ldots(5)$$

The equation of the required plane is obtained by eliminating A, B, C between the equations (3), (4) and (5). Hence the equation of the required plane is

$$\begin{vmatrix} x & y & z \\ bc'-b'c & ca'-c'a & ab'-a'b \\ \beta\gamma'-\beta'\gamma & \gamma\alpha'-\gamma'\alpha & \alpha\beta'-\alpha'\beta \end{vmatrix}=0.$$

Ex. 10. *Find the equation of the plane through the point* $(2, -1, 1)$ *and the line* $4x-3y+5=0=y-2z-5.$

Sol. The equations of the line are

$$4x-3y+5=0,\ y-2z-5=0.$$

The equation of any plane through the given line [using $P+\lambda Q=0$] is $4x-3y+5+\lambda\,(y-2z-5)=0. \quad \ldots(1)$

If the plane (1) passes through the point $(2, -1, 1)$, we have

$$4(2)-3(-1)+5+\lambda\,\{-1-2.1-5\}=0$$

or $16-8\lambda=0$ or $\lambda=2.$

Putting the value of λ in (1), the equation of the required plane is

$$4x-3y+5+2\,(y-2z-5)=0 \text{ or } 4x-y-4z=5.$$

Ex. 11. *Prove that the equation to the two planes inclined at an angle α to xy-plane and containing the line $y=0$, $z \cos \beta = x \sin \beta$ is*

$$(x^2+y^2) \tan^2 \beta + z^2 - 2zx \tan \beta = y^2 \tan^2 \alpha. \quad \textbf{(M.U. 1990P)}$$

Sol. The equation of any plane through the line $y=0$, $z \cos \beta = x \sin \beta$ is given by

$$(x \sin \beta - z \cos \beta) + \lambda y = 0. \quad \ldots(1)$$

The other plane is the xy-plane, whose equation is $z=0$ *i.e.*

$$0.x + 0.y + 1.z = 0. \quad \ldots(2)$$

The d.r.'s of the normal to the plane (1) are $\sin \beta$, λ, $-\cos \beta$, and the d.r.'s of the normal to the plane (2) are 0, 0, 1. Also the angle between them is α and hence we have

$$\cos \alpha = \frac{\sin \beta . 0 + \lambda . 0 - \cos \beta . 1}{\sqrt{(\sin^2 \beta + \lambda^2 + \cos^2 \beta)} . \sqrt{(0+0+1)}}$$

or $\cos \alpha \sqrt{(\sin^2 \beta + \cos^2 \beta + \lambda^2)} = -\cos \beta$

or $\cos \alpha \sqrt{(1+\lambda^2)} = -\cos \beta$

or $(\lambda^2+1) \cos^2 \alpha = \cos^2 \beta$

or $\lambda^2 \cos^2 \alpha = \cos^2 \beta - \cos^2 \alpha$

or $\lambda = \pm \sqrt{(\cos^2 \beta - \cos^2 \alpha)} / \cos \alpha.$

It gives two values of λ and hence there are two planes containing the given line and inclined at an angle α to the xy-plane. Substituting these values of λ in (1) and multiplying both the equations thus obtained the combined equation of the two required planes is given by

$$\left\{ x \sin \beta - z \cos \beta + \frac{\sqrt{(\cos^2 \beta - \cos^2 \alpha)}}{\cos \alpha} y \right\}$$

$$\left\{ x \sin \beta - z \cos \beta - \frac{\sqrt{(\cos^2 \beta - \cos^2 \alpha)}}{\cos \alpha} y \right\} = 0$$

or $(x \sin \beta - z \cos \beta)^2 - \dfrac{\cos^2 \beta - \cos^2 \alpha}{\cos^2 \alpha} y^2 = 0$

or $x^2 \sin^2 \beta + z^2 \cos^2 \beta - 2xz \sin \beta \cos \beta + y^2 = y^2 \cos^2 \beta \sec^2 \alpha.$

Dividing throughout by $\cos^2 \beta$, the above equation becomes

$$x^2 \tan^2 \beta + z^2 - 2xz \tan \beta + y^2 \sec^2 \beta = y^2 \sec^2 \alpha$$

or $x^2 \tan^2 \beta + z^2 - 2zx \tan \beta + y^2 (1+\tan^2 \beta) = y^2 (1+\tan^2 \alpha)$

or $(x^2+y^2) \tan^2 \beta + z^2 - 2zx \tan \beta = y^2 \tan^2 \alpha.$

Ex. 12. *The plane $lx+my=0$ is rotated about its line of intersection with the plane $z=0$, through an angle α. Prove that the equation of the plane in its new position is*

$$lx + my \pm z\sqrt{(l^2+m^2)} \tan \alpha = 0.$$

Sol. The equations of the given planes are

$$lx+my=0 \quad \ldots(1) \quad \text{and} \quad z=0. \quad \ldots(2)$$

The equation of any plane through the line of intersection of the planes (1) and (2) is

$$lx+my+\lambda z=0. \quad \ldots(3)$$

Suppose the plane (1) when rotated through an angle α about its line of intersection with the plane $z=0$ has the equation (3). Thus the angle between the planes (1) and (3) is α.

$$\therefore \quad \cos \alpha = \frac{l \, l + m \, m + 0.\lambda}{\sqrt{(l^2+m^2)} \sqrt{(l^2+m^2+\lambda^2)}}$$

or $\cos \alpha = \dfrac{\sqrt{(l^2+m^2)}}{\sqrt{(l^2+m^2+\lambda^2)}}$ or $\cos^2 \alpha = \dfrac{l^2+m^2}{l^2+m^2+\lambda^2}$

or $(l^2+m^2)(1-\cos^2 \alpha) = \lambda^2 \cos^2 \alpha$

or $\lambda^2 = (l^2+m^2) \tan^2 \alpha, \quad \lambda = \pm \sqrt{(l^2+m^2)} \tan \alpha.$

Putting this value of λ in (3), the required equation of the plane in its new position is given by

$$lx+my \pm (l^2+m^2).z \tan \alpha = 0.$$

§ 7. Foot and length of perpendicular from a point to a line.

(A) Line in symmetrical form.

To find the equations and length of the perpendicular distance of a point $P(x_1, y_1, z_1)$ from a given line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

The equations of the given line in symmetrical form are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)}. \quad \ldots(1)$$

The co-ordinates of any point N on the line (1) are

$$(\alpha+lr, \beta+mr, \gamma+nr). \quad \ldots(2)$$

Let this point N be the foot of the perpendicular from the point $P(x_1, y_1, z_1)$ to the line (1), so that the line PN is perpendicular to the line (1).

The direction ratios of the line PN are given by

$$\alpha+lr-x_1, \beta+mr-y_1, \gamma+nr-z_1. \quad \ldots(3)$$

Since PN is perpendicular to the line (1), using the condition $a_1a_2+b_1b_2+c_1c_2=0$, we have

$$l(\alpha+lr-x_1)+m(\beta+mr-y_1)+n(\gamma+nr-z_1)=0$$

or $r(l^2+m^2+n^2) = l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)$

or $r = [l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]/(l^2+m^2+n^2).$ $\ldots(4)$

$\therefore$ The equations of the perpendicular from the point $P(x_1, y_1, z_1)$ to the line (1) are given by

$$\frac{x-x_1}{\alpha+lr-x_1}=\frac{y-y_1}{\beta+mr-y_1}=\frac{z-z_1}{\gamma+nr-z_1} \quad ...(5)$$

where r is given by (4).

Also substituting the value of r from (4) in (2), the co-ordinates of the foot N of the perpendicular are determined, and then the perpendicular distance PN is easily calculated. See Ex. 1 which follows after part (B) of this article.

(B) Line in general form :

To find the equations of the perpendicular line from the point $P(x_1, y_1, z_1)$ to a line whose equations are given by

$$ax+by+cz+d=0=a'x+b'y+c'z+d'.$$

The equations of the given line in general form are

$$ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0. \quad ...(1)$$

The perpendicular from a point P to the given line **(1)** *is the intersection of the two planes namely (i) the plane through the given point P and also through the given line and (ii) the plane through the point P perpendicular to the given line.* **[Remember]**

Let l, m, n be the d.c.'s of the given line (1), then we have

$$al+bm+cn=0, \quad \text{and} \quad a'l+b'm+c'n=0.$$

Solving these, we have

$$\frac{l}{bc'-b'c}=\frac{m}{ca'-c'a}=\frac{n}{ab'-a'b}. \quad ...(2)$$

Now the equation of any plane through the line (1) is given by

$$(ax+by+cz+d)+\lambda\,(a'x+b'y+c'z+d')=0. \quad ...(3)$$

If the plane (3) also passes through the point $P(x_1, y_1, z_1)$

then $(ax_1+by_1+cz_1+d)+\lambda\,(a'x_1+b'y_1+c'z_1+d')=0$

or $\lambda=-(ax_1+by_1+cz_1+d)/(a'x_1+b'y_1+c'z_1+d')$.

Substituting this value of λ in (3), the equation of the plane through the point P and the line (1) is given by

$$\frac{ax+by+cz+d}{ax_1+by_1+cz_1+d}=\frac{a'x+b'y+c'z+d'}{a'x_1+b'y_1+c'z_1+d'}; \quad ...(4)$$

Now we are to find the equation of the second plane which passes through P and is perpendicular to the line (1).

Since the plane is perpendicular to the line (1), therefore the d.r.'s of its normal are proportional to l, m, n given by (2).

Therefore the equation of the plane perpendicular to the line (1) and passing through $P(x_1, y_1, z_1)$ is

$$l\,(x-x_1)+m\,(y-y_1)+n\,(z-z_1)=0. \quad ...(5)$$

Therefore the equations of the perpendicular line from the point $P(x_1, y_1, z_1)$ to the line (1) are given by the equations (4) and (5).

SOLVED EXAMPLES (E)

Ex. 1. *Find the equations of the perpendicular from the point $(3, -1, 11)$ to the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$. Find also the co-ordinates of the foot of the perpendicular. Hence find the length of the perpendicular.*

Sol. The given point is $P(3, -1, 11)$ and the equations of the given line are

$$\frac{x-0}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r \text{ (say)}. \quad ...(1)$$

The co-ordinates of any point N on the line (1) are

$$(2r, 3r+2, 4r+3). \quad ...(2)$$

Let this point N be the foot of the perpendicular from the point $P(3, -1, 11)$ to the line (1).

Then the d.r.'s of the perpendicular PN are

$$2r-3,\ (3r+2)-(-1),\ (4r+3)-11,$$

or $$2r-3,\ 3r+3,\ 4r-8. \quad ...(3)$$

The d.r.'s of the given line (1) are 2, 3, 4. Now PN is perpendicular to the line (1). The condition of perpendicularity gives

$$(2r-3).2+(3r+3).3+(4r-8).4=0,$$

or $29r-29=0$, or $r=1$.

Putting the value of r in (2), the foot N of the perpendicular is the point $(2, 5, 7)$.

Putting the value of r in (3), the d.r.'s of PN are $-1, 6, -4$. Hence the equations of the perpendicular PN from the point $P(3, -1, 11)$ to the line (1) are

$$\frac{x-3}{-1}=\frac{y+1}{6}=\frac{z-11}{-4}.$$

The length of the perpendicular PN = the distance between the points $P(3, -1, 11)$ and $N(2, 5, 7)$

$$=\sqrt{[(3-2)^2+(-1-5)^2+(11-7)^2]}$$
$$=\sqrt{(1+36+16)}=\sqrt{(53)}.$$

Ex. 2. *Find the equations of the perpendicular from origin to the line*

$$ax+by+cz+d=0=a'x+b'y+c'z+d'=0.$$

Sol. First read § 7 (B) again and then solve this problem.

The equations of the given line are

$$ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0. \quad \ldots(1)$$

We know that the perpendicular from a given point P to a given line is the intersection of the two planes, namely (i) the plane through the given point $P(0, 0, 0)$ and also through the given line and (ii) the plane through the point P perpendicular to the given line. Thus we proceed as follows :

The equation of any plane through the line (1) is

$$ax+by+cz+d+\lambda(a'x+b'y+c'z+d')=0. \quad \ldots(2)$$

If the plane (2) passes through the point $P(0, 0, 0)$, then

$$d+\lambda d'=0, \text{ or } \lambda=-d/d'.$$

Putting this value of λ in (2), the equation of the plane through the origin and the given line is

$$(ax+by+cz+d)-(d/d')(a'x+b'y+c'z+d')=0$$

or $$(ad'-a'd)x+(bd'-b'd)y+(cd'-c'd)z=0. \quad \ldots(3)$$

Now let l, m, n be the d.r.'s of the given line (1), then

$$al+bm+cn=0, \text{ and } a'l+b'm+c'n=0.$$

Solving, $$\frac{l}{bc'-b'c}=\frac{m}{ca'-c'a}=\frac{n}{ab'-a'b}. \quad \ldots(4)$$

Thus the equation of the plane through $P(0, 0, 0)$ and perpendicular to the line (1) is

$$l(x-0)+m(y-0)+n(z-0)=0$$

or $$(bc'-b'c)x+(ca'-c'a)y+(ab'-a'b)z=0. \quad \ldots(5)$$

Hence (3) and (5) together are the equations of the perpendicular from the origin to the given line (1).

Ex 3. *Show that the distance d of the point $P(\alpha, \beta, \gamma)$ from the line $(x-x_1)/l=(y-y_1)/m=(z-z_1)/n$ measured parallel to the plane* $ax+by+cz+d=0$ *is given by*

$$d^2=\frac{(a^2+b^2+c^2)\,\Sigma\{m(z_1-\gamma)-n(y_1-\beta)\}^2-[\Sigma(x_1-\alpha)(bn-cm)]^2}{(al+bm+cn)^2}.$$

Sol. The equations of the given line are

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n=r, \text{ (say)}. \quad \ldots(1)$$

Any point Q on (1) is

$$(x_1+lr,\ y_1+mr,\ z_1+nr).$$

Now P is the point (α, β, γ) and hence d.r.'s of PQ are

$$x_1+lr-\alpha,\ y_1+mr-\beta,\ z_1+nr-\gamma.$$

It is required to find the distance PQ measured parallel to the plane $ax+by+cz+d=0$. Now PQ is parallel to this plane and hence PQ will be perpendicular to the normal to the plane

$$ax+by+cz+d=0.$$

Therefore we have

$$(lr+x_1-\alpha)a+(y_1+mr-\beta)b+(z_1+nr-\gamma)c=0$$

or $$r=-[a(x_1-\alpha)+b(y_1-\beta)+c(z_1-\gamma)]/(al+bm+cn). \quad \ldots(2)$$

Now $$d^2=PQ^2=(x_1+lr-\alpha)^2+(y_1+mr-\beta)^2+(z_1+nr-\gamma)^2$$

or $$d^2=\{lr+(x_1-\alpha)\}^2+\{mr+(y_1-\beta)\}^2+\{nr+(z_1-\gamma)\}^2$$

or $$d^2=r^2(l^2+m^2+n^2)+2r\{l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)\}$$
$$+\{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2\}. \quad \ldots(3)$$

Putting the value of r from (2) in (3) and using Lagrange's identity, we get the required result.

Ex. 4. *Find the distance of the point $P(3, 8, 2)$ from the line* $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}$ *measured parallel to the plane*

$$3x+2y-2z+17=0.$$

Sol. The equations of the given line are

$$(x-1)/2=(y-3)/4=(z-2)/3=r, \text{ (say)}. \quad \ldots(1)$$

Any point Q on the line (1) is

$$(2r+1,\ 4r+3,\ 3r+2).$$

Now P is the point $(3, 8, 2)$ and hence d.r.'s of PQ are

$$2r+1-3,\ 4r+3-8,\ 3r+2-2 \text{ i.e. } 2r-2,\ 4r-5,\ 3r.$$

It is required to find the distance PQ measured parallel to the plane

$$3x+2y-2z+17=0. \quad \ldots(2)$$

Now PQ is parallel to the plane (2) and hence PQ will be perpendicular to the normal to the plane (2). Hence we have

$$(2r-2)(3)+(4r-5)(2)+(2r)(-2)=0,$$

or $$8r-16=0, \text{ or } r=2.$$

Putting the value of r, the point Q is $(5, 11, 8)$.

$\therefore$ Required distance = The distance between $P(3, 8, 2)$ and $Q(5, 11, 8)$

$$=\sqrt{[(3-5)^2+(8-11)^2+(2-8)^2]}=\sqrt{(4+9+36)}=7.$$

Ex. 5. *The equations to AB referred to rectangular axes are $x/2=y/-3=z/6$. Through a point $P(1, 2, 5)$, PN is drawn perpendicular to AB and PQ is drawn parallel to the plane $3x+4y+5z=0$*

to meet AB in Q. *Find the equations to PN and PQ and the co-ordinates of N and Q.*

Sol. The equations of the line AB are given as

$$x/2=y/-3=z/6=r \text{ (say)}. \quad ...(1)$$

Any point on (1) is $(2r, -3r, 6r)$. Let this point be N. Also P is $(1, 2, 5)$.

$\therefore$ The d.r.'s of PN are $2r-1, -3r-2, 6r-5$.

Now PN will be perpendicular to the line AB given by the equations (1) if

$$2(2r-1)+(-3)(-3r-2)+6(6r-5)=0, \text{ or } r=\tfrac{26}{49}.$$

Putting the value of r, we have $N\left(\frac{52}{49}, \frac{-78}{49}, \frac{156}{49}\right)$

and d.r.'s of PN are $\frac{52}{49}-1, \frac{-78}{49}-2, \frac{156}{49}-5$

or $\quad 3, -176, -89.$

Thus the equations to PN are

$$\frac{x-1}{3}=\frac{y-2}{-176}=\frac{z-5}{-89}. \quad ...(2)$$

Again the co-ordinates of any point Q on the line (1) are $(2r, -3r, 6r)$ and the point P is $(1, 2, 5)$.

$\therefore$ The d.r.'s of PQ are $2r-1, -3r-2, 6r-5$.

Now PQ is drawn parallel to the plane

$$3x+4y+5z=0. \quad ...(3)$$

$\therefore$ The line PQ is perpendicular to the normal to the plane (3), whose d.r.'s are 3, 4, 5.

Therefore we have

$$3(2r-1)+4(-3r-2)+5(6r-5)=0, \text{ or } r=\tfrac{3}{2}.$$

Putting this value of r, we have $Q(3, -9/2, 9)$ and d.r.'s of PQ are $3-1, (-9/2)-2, 9-5$ *i.e.* $4, -13, 8$.

$\therefore$ The equations of PQ are given by

$$\frac{x-1}{4}=\frac{y-2}{-13}=\frac{z-5}{8}.$$

The projection (or image) of a line on (or in) a given plane.

The method is explained by the following examples.

The line is in symmetrical form.

Ex. 6. *What do you understand by the projection of a line on a given plane? Find the equations of the projection of the line* $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ *on the plane* $x+2y+z=6$.

Solution. Definition 1. *The projection of a line on a given plane is the line of intersection of the two planes namely (i) the given plane and (ii) the plane through the given line and perpendicular to the given plane.*

Definition 2. *Let P be the point of intersection of the given line with the given plane and let Q be the foot of the perpendicular from any point on the line to the plane, then the line PQ is said to be the projection of the given line on the given plane.*

Now we give the solution of the given problem using both definitions one by one.

Using definition 1.

The equations of the given line are

$$\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4} \quad ...(1)$$

and the equation of the given plane is $x+2y+z=6$. ...(2)

The equation of any plane through the given line (1) is

$$A(x-1)+B(y+1)+C(z-3)=0 \quad ...(3)$$

where $\quad 2A-B+4C=0. \quad ...(4)$

The plane (3) will be perpendicular to the plane (2), if

$$A+2B+C=0. \quad ...(5)$$

Solving (4) and (5), we get $\frac{A}{-9}=\frac{B}{2}=\frac{C}{5}$.

Putting these proportionate values of A, B, C in (3), we have

$$9(x-1)+2(y+1)+5(z-3)=0$$

or $\quad 9x-2y-5z+4=0. \quad ...(6)$

The equations (2) and (6) together are the equations of the line of projection.

Using definition 2.

The co-ordinates of any point on the line (1) are $(2r+1, -r-1, 4r+3)$. Let this be the point of intersection P of the given line (1) with the given plane (2). Then $P(2r+1, -r-1, 4r+3)$ will lie on the plane (2).

$$\therefore \quad 2r+1+2(-r-1)+4r+3=6, \text{ or } r=1.$$

Putting this value of r, the point P is $(3, -2, 7)$.

Now the line (1) clearly passes through the point $(1, -1, 3)$ and hence this is a point on the given line (1). Now we are to find the foot of the perpendicular Q from $(1, -1, 3)$ to the plane (2).

The d.r.'s of the normal to the plane (2) are 1, 2, 1 and hence these are d.r.'s of the line through (1, −1, 3) and perpendicular to the plane (2), and therefore equations of this perpendicular line are

$$\frac{x-1}{1}=\frac{y+1}{2}=\frac{z-3}{1}=r_1 \text{ (say)}.$$

Any point on it is $(r_1+1, 2r_1-1, r_1+3)$. Let this be the point Q and so it will lie on the plane (2).

$\therefore$ $r_1+1+2(2r_1-1)+r_1+3=6$, or $r_1=\frac{2}{3}$.

Putting this value of r_1, the foot of the perpendicular Q is

$$(5/3, 1/3, 11/3).$$

$\therefore$ The equations of the projection *i.e.* the equations of the line PQ joining the points $P(3, -2, 7)$ and $Q(\frac{5}{3}, \frac{1}{3}, \frac{11}{3})$ are

$$\frac{x-3}{3-\frac{5}{3}}=\frac{y+2}{-2-\frac{1}{3}}=\frac{z-7}{7-\frac{11}{3}} \text{ or } \frac{x-3}{4}=\frac{y+2}{-7}=\frac{z-7}{10}. \quad ...(7)$$

Remark. If the equations (2) and (6) are transformed to symmetrical form, we shall get the equations (7) of the projection PQ. We note that if we want to find the equations of the projection in general form then we use definition one, and if we want symmetrical form then we use definition two.

Ex. 7. *If L is the line* $\frac{x-1}{2}=\frac{y}{-1}=\frac{z+2}{1}$, *find the direction cosines of the projection of L on the plane* $2x+y-3z=4$ *and the equation of the plane through L parallel to the line*

$$2x+5y+3z=4,\ x-y-3z=6.$$

Sol. The equations of the line L are

$$\frac{x-1}{2}=\frac{y-0}{-1}=\frac{z+2}{1}=r, \text{ (say)}. \quad ...(1)$$

Any point on the line (1) is $(2r+1, -r, r-2)$.

If it lies on the given plane $2x+y-3z=4$, ...(2)

we have $2(2r+1)+(-r)-3(r-2)=4$, or $0.r+4=0$.

This relation is impossible and we do not get any value of r as the coefficient of r is zero. This shows that line (1) is parallel to the plane (2). Hence method of definition 2 (see Ex. 8) cannot be used here. Now we shall use **method of definition 1.**

The equation of any plane through the line (1) is

$$A(x-1)+By+C(z+2)=0, \quad ...(3)$$

where $2A-B+C=0$. ...(4)

The plane (3) will be perpendicular to the plane (2) if

$$2A+B-3C=0. \quad ...(5)$$

Solving (4) and (5), we get

$$\frac{A}{2}=\frac{B}{8}=\frac{C}{4} \text{ or } \frac{A}{1}=\frac{B}{4}=\frac{C}{2}.$$

Putting these proportionate values of A, B and C in (3), the equation of the plane through the line L (1) and perpendicular to the plane (2) is given by

$$1.(x-1)+4.y+2(z+2)=0$$

or $x+4y+2z+3=0$. ...(6)

The equations (2) and (6) together are the equations of the line of projection of the line L on the plane (2).

Let l, m, n be the d.c.'s of the line of projection given by (2) and (6). Then we have

$$2l+m-3n=0 \text{ and } l+4m+2n=0.$$

Solving these, we get $\frac{l}{14}=\frac{m}{-7}=\frac{n}{7}$

or $\frac{l}{2}=\frac{m}{-1}=\frac{n}{1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(2)^2+(-1)^2+(1)^2\}}}=\frac{1}{\sqrt{6}}.$

$\therefore$ $l=2/\sqrt{6}$, $m=-1/\sqrt{6}$, $n=1/\sqrt{6}$.

Hence d.c.'s of the line of projection are $2/\sqrt{6}$, $-1/\sqrt{6}$, $1/\sqrt{6}$.

Now to find the equation of the plane through the line L and parallel to the line

$$2x+5y+3z=4,\ x-y-5z=6. \quad ...(7)$$

Let l_1, m_1, n_1 be the d.c.'s of the line (7).

Then $2l_1+5m_1+3n_1=0$, $l_1-m_1-5n_1=0$.

Solving these we get $\frac{l_1}{22}=\frac{m_1}{-13}=\frac{n_1}{7}$.

Now the equation of any plane through the line L (1) is given by (3) provided the condition (4) holds. If the plane (3) is parallel to the line (7), then the normal to the plane (3) will be perpendicular to the line (7), the condition for which is

$$22A-13B+7C=0. \quad ...(8)$$

Solving (4) and (8), we get

$$\frac{A}{-7+13}=\frac{B}{22-14}=\frac{C}{-26+22} \text{ or } \frac{A}{3}=\frac{B}{4}=\frac{C}{-2}.$$

Substituting these proportionate values of A, B and C in (3), the equation of the plane through the line L (1) parallel to the line (7) is given by

$$3(x-1)+4y-2(z+2)=0, \text{ or } 3x+4y-2z=7.$$

The line in general form.

Ex. 8. *Find the projection of the line* $3x-y+2z=1$, $x+2y-z=2$ *on the plane* $3x+2y+z=0$.

Sol. The equations of the given line are

$$3x-y+2z=1,\ x+2y-z=2 \quad \ldots(1)$$

The equation of the given plane is $3x+2y+z=0$. ...(2)

The equation of any plane through the line (1) is

$$(3x-y+2z-1)+\lambda\,(x+2y-z-2)=0$$

or
$$(3+\lambda)\,x+(-1+2\lambda)\,y+(2-\lambda)\,z-1-2\lambda=0. \quad \ldots(3)$$

The plane (3) will be perpendicular to the plane (2), if

$$3\,(3+\lambda)+2\,(-1+2\lambda)+1\,(2-\lambda)=0, \text{ or } \lambda=-\tfrac{3}{2}.$$

Putting this value of λ in (3), the equation of the plane through the line (1) and perpendicular to the plane (2) is given by

$$(3-\tfrac{3}{2})\,x+(-1-3)\,y+(2+\tfrac{3}{2})\,z-1+3=0$$

or
$$3x-8y+7z+4=0. \quad \ldots(4)$$

$\therefore$ The projection of the given line (1) on the given plane (2) is given by the equations (2) and (4) together.

Note. The symmetrical form of the projection given above by equations (2) and (4) is

$$\frac{x+\frac{4}{5}}{-11}=\frac{y-\frac{2}{5}}{9}=\frac{z}{15}.$$

§ 8. **Coplanar Lines.** *To find the condition that the two lines whose equations are given may be coplanar i.e. should intersect and to obtain the equation of the plane containing them.*

(Gorakhpur 1982)

Two lines are called coplanar if they intersect. Even if the two lines are parallel, they are also coplanar because they will intersect at infinity and will satisfy the condition of coplanarity given below.

The equations of two lines may be given in three ways :

(A) Both lines in symmetrical form, (B) one line in symmetrical form and the other in general form, and (C) both lines in general form.

Now we shall discuss these cases one by one.

(A) Both lines are given in symmetrical form :

(Avadh 1981, 82; Allahabad 80; Kanpur 76, 81)

Let the equations of the two given lines be

$$(x-x_1)/l_1=(y-y_1)/m_1=(z-z_1)/n_1, \quad \ldots(1)$$

and
$$(x-x_2)/l_2=(y-y_2)/m_2=(z-z_2)/n_2. \quad \ldots(2)$$

The equation of any plane through the line (1) is

$$A\,(x-x_1)+B\,(y-y_1)+C\,(z-z_1)=0, \quad \ldots(3)$$

where
$$Al_1+Bm_1+Cn_1=0. \quad \ldots(4)$$

For the lines (1) and (2) to be coplanar, the plane (3) should be such that the line (2) also lies in this plane. Then the normal to the plane (3) will be perpendicular to the line (2) so that we have

$$Al_2+Bm_2+Cn_2=0. \quad \ldots(5)$$

Again the point (x_2, y_2, z_2) through which the line (2) passes will also lie on the plane (3) so that we have

$$A(x_2-x_1)+B(y_2-y_1)+C(z_2-z_1)=0. \quad \ldots(6)$$

Hence the required condition that the lines (1) and (2) are coplanar (*i.e.* are intersecting) is obtained by eliminating A, B, C from the relations (6), (4) and (5), and is given by

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0. \quad \ldots\textbf{(A)}$$

The equation of the plane containing the lines (1) and (2) is obtained by eliminating A, B, C between (3), (4) and (5) and is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

The point of intersection of the lines (1) and (2).

The co-ordinates of any point P on the line (1) are $(l_1r_1+x_1, m_1r_1+y_1, n_1r_1+z_1)$ and those of a point Q on the line (2) are $(l_2r_2+x_2, m_2r_2+y_2, n_2r_2+z_2)$.

If the lines (1) and (2) intersect, then for some values of r_1 and r_2, the points P and Q should coincide, *i.e.* we have

$$l_1r_1+x_1=l_2r_2+x_2,\ m_1r_1+y_1=m_2r_2+y_2,\ n_1r_1+z_1=n_2r_2+z_2$$

or
$$(x_1-x_2)+l_1r_1-l_2r_2=0, \quad \ldots(7)$$

$$(y_1-y_2)+m_1r_1-m_2r_2=0 \quad \ldots(8)$$

and
$$(z_1-z_2)+n_1r_1-n_2r_2=0. \quad \ldots(9)$$

Solving the relations (7) and (8) the values of r_1 and r_2 are obtained and if these values also satisfy the relation (9), then the lines (1) and (2) are coplanar otherwise not. In case the lines (1) and (2) are coplanar, on substituting the value of r_1 (or r_2) the co-ordinates of the point of intersection P (or Q) are obtained.

Also by eliminating r_1 and $-r_2$ between the relations (7), (8) and (9) the condition that the lines (1) and (2) are coplanar is given by

$$\begin{vmatrix} x_1-x_2 & l_1 & l_2 \\ y_1-y_2 & m_1 & m_2 \\ z_1-z_2 & n_1 & n_2 \end{vmatrix}=0$$

or
$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

(B) **One line is given in symmetrical form and the other line is given in general form.**

(Agra 1973 ; Meerut 82S ; Allahabad 76, 81)

Let the equations of the two given lines be

$$(x-x_1)/l=(y-y_1)/m=(z-z_1)/n \quad ...(1)$$

and $$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2. \quad ...(2)$$

The equation of any plane through the line (2) is

$$(a_1x+b_1y+c_1z+d_1)+\lambda\,(a_2x+b_2y+c_2z+d_2)=0 \quad ...(3)$$

or $$(a_1+\lambda a_2)\,x+(b_1+\lambda b_2)\,y+(c_1+\lambda c_2)\,z+(d_1+\lambda d_2)=0. \quad ...(3')$$

If the line (1) is parallel to the plane (3′), then the normal of the plane (3′) will be perpendicular to the line (1) and so we have $l\,(a_1+\lambda a_2)+m\,(b_1+\lambda b_2)+n\,(c_1+\lambda c_2)=0$

or $\lambda\,(a_2l+b_2m+c_2n)=-(a_1l+b_1m+c_1n)$

or $$\lambda=-(a_1l+b_1m+c_1n)/(a_2l+b_2m+c_2n). \quad ...(4)$$

Putting this value of λ in (3), the equation of the plane through the line (2) and parallel to the line (1) is given by

$$\frac{a_1x+b_1y+c_1z+d_1}{a_1l+b_1m+c_1n}=\frac{a_2x+b_2y+c_2z+d_2}{a_2l+b_2m+c_2n}. \quad ...(5)$$

The line (1) passes through the point (x_1, y_1, z_1) which must lie on the plane (5) if the lines (1) and (2) are coplanar *i.e.*, intersecting. Thus for the lines (1) and (2) to be coplanar the point (x_1, y_1, z_1) should satisfy the equation (5). Hence the condition that the lines (1) and (2) are coplanar is given by

$$\frac{a_1x_1+b_1y_1+c_1z_1+d_1}{a_1l+b_1m+c_1n}=\frac{a_2x_1+b_2y_1+c_2z_1+d_2}{a_2l+b_2m+c_2n}. \quad ...(6)$$

If the condition (6) is satisfied the lines (1) and (2) are intersecting (or coplanar) and the plane containing both the lines is given by the equation (5).

Alternative Method : The co-ordinates of any point P on the line (1) are $(lr+x_1, mr+y_1, nr+z_1)$. If the lines (1) and (2) are intersecting then the point P should also lie on the line (2). Hence we have

$$a_1\,(lr+x_1)+b_1\,(mr+y_1)+c_1\,(nr+z_1)+d_1=0,$$
and $$a_2\,(lr+x_1)+b_2\,(mr+y_1)+c_2\,(nr+z_1)+d_2=0.$$

From these, we get

$$r\,(a_1l+b_1m+c_1n)=-(a_1x_1+b_1y_1+c_1z_1+d_1),$$
and $$r\,(a_2l+b_2m+c_2n)=-(a_2x_1+b_2y_1+c_2z_1+d_2).$$

Eliminating r, the required condition that the lines (1) and (2) are coplanar is given by

$$\frac{a_1x_1+b_1y_1+c_1z_1+d_1}{a_1l+b_1m+c_1n}=\frac{a_2x_1+b_2y_1+c_2z_1+d_2}{a_2l+b_2m+c_2n}$$

(C) **Both the lines are given in general form :**

Let the equations of the two lines be

$$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2, \quad ...(1)$$

and $$a_3x+b_3y+c_3z+d_3=0=a_4x+b_4y+c_4z+d_4. \quad ...(2)$$

If the lines (1) ond (2) are coplanar, these will intersect in a point say the point (α, β, γ).

The co-ordinates of this point must satisfy the equations of the four planes representing the two lines, and so we have

$$a_1\alpha+b_1\beta+c_1\gamma+d_1=0,$$
$$a_2\alpha+b_2\beta+c_2\gamma+d_2=0,$$
$$a_3\alpha+b_3\beta+c_3\gamma+d_3=0,$$
and $$a_4\alpha+b_4\beta+c_4\gamma+d_4=0.$$

Eliminating α, β, γ between the above four relations, the required condition is given by

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}=0$$

Remark. The expansion of a fourth order determinant is usually difficult and therefore it is convenient to solve numerical examples by first reducing the equations of the lines to symmetrical forms and then proceeding as in case (A).

SOLVED EXAMPLES (F)

Ex. 1. *Show that the lines* $\frac{1}{2}(x+3)=\frac{1}{3}(y+5)=-\frac{1}{3}(z-7)$ *and* $\frac{1}{4}(x+1)=\frac{1}{5}(y+1)=-(z+1)$ *are coplanar. Find the equation of the plane containing them.* **(Rohilkhand 1982; Madras 76)**

Sol. The equations of the given lines are

$$(x+3)/2=(y+5)/3=(z-7)/-3=r_1 \text{ (say)}, \quad \ldots(1)$$

and

$$(x+1)/4=(y+1)/5=(z+1)/-1=r_2 \text{ (say)}. \quad \ldots(2)$$

The co-ordinates of any point P on the line (1) are $(2r_1-3, 3r_1-5, -3r_1+7)$ and those of any point Q on the line (2) are $(4r_2-1, 5r_2-1, -r_2-1)$.

If the lines (1) and (2) are coplanar then they intersect and hence for some values of r_1 and r_2 the points P and Q coincide. Thus we have

$$2r_1-3=4r_2-1,\ 3r_1-5=5r_2-1,\ -3r_1+7=-r_2-1$$

or

$$r_1-2r_2=1,\ 3r_1-5r_2=4,\ 3r_1-r_2=8.$$

Solving the first two equations, we get $r_1=3$, $r_2=1$.

These values of r_1 and r_2 also satisfy the third equation, and hence the lines (1) and (2) are coplanar (*i.e.* intersect).

Putting the value of r_1 (or the value of r_2) the co-ordinates of the common point of intersection *i.e.*, P (or Q) are $(3, 4, -2)$.

Now the equation of the plane containing the lines (1) and (2) [*i.e.* the plane containing the line (1) and parallel to the line (2)] is

$$\begin{vmatrix} x+3 & y+5 & z-7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix}=0 \quad \text{[See § 8 (A)]}$$

or $(x+3)\{3(-1)-(-3)5\}-(y+5)\{2(-1)-(-3)4\}+(z-7)\{2.5-3.4\}=0$

or $(x+3).12-(y+5).10+(z-7).(-2)=0$ or $6x-5y-z=0$.

Ex. 2. *Prove that the lines* $\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$ *and* $\frac{1}{3}(x-2)=\frac{1}{4}(y-3)=\frac{1}{5}(z-4)$ *are coplanar; find their point of intersection.* **(Berahmpur 1981S ; Lucknow 81 ; Meerut 75)**

Also find the equation of the plane in which they lie.

Sol. The equations of the given lines are

$$(x-1)/2=(y-2)/3=(z-3)/4=r_1 \text{ (say)}. \quad \ldots(1)$$

and

$$(x-2)/3=(y-3)/4=(z-4)/5=r_2 \text{ (say)}. \quad \ldots(2)$$

The co-ordinates of any point P on the line (1) are $(2r_1+1, 3r_1+2, 4r_1+3)$ and those of any point Q on the line (2) are $(3r_2+2, 4r_2+3, 5r_2+4)$.

If the lines (1) and (2) are coplanar then they intersect and hence for some values of r_1 and r_2 the points P and Q coincide. Thus we have

$$2r_1+1=3r_2+2, \text{ or } 2r_1-3r_2=1 \quad \ldots(3)$$

$$3r_1+2=4r_2+3, \text{ or } 3r_1-4r_2=1 \quad \ldots(4)$$

and

$$4r_1+3=5r_2+4, \text{ or } 4r_1-5r_2=1. \quad \ldots(5)$$

Solving (3) and (5), we get $r_1=-1$, $r_2=-1$.

These values of r_1 and r_2 also satisfy the equation (4), and hence the given lines (1) and (2) are coplanar. Putting the values of r_1 (or of r_2), the point of intersection P (or Q) is $(-1, -1, -1)$.

Now the equation of the plane in which the lines (1) and (2) lie is given by

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=0 \quad \text{[See § 8 (A)]}$$

or

$$x-2y+z=0.$$

Ex. 3. *Show that the lines*

$$(x+1)/(-3)=(y-3)/2=(z+2)/1$$

and

$$x/1=(y-7)/(-3)=(z+7)/2$$

intersect. Find the co-ordinates of the point of intersection and the equation to the plane containing them.

[Agra 1979, Gorakhpur 82, Kanpur 81, Rohilkhand 77]

Sol. The equations of the given lines are

$$(x+1)/(-3)=(y-3)/2=(z+2)/1=r_1 \text{ (say)} \quad \ldots(1)$$

and

$$(x-0)/1=(y-7)/(-3)=(z+7)/2=r_2 \text{ (say)} \quad \ldots(2)$$

The co-ordinates of any point P on the line (1) are $(-3r_1-1, 2r_1+3, r_1-2)$ and those of any point Q on the line (2) are $(r_2, -3r_2+7, 2r_2-7)$.

If the lines (1) and (2) intersect (*i.e.* are coplanar), then for some values of r_1 and r_2 the points P and Q coincide. Thus, we have

$$-3r_1-1=r_2, \text{ or } 3r_1+r_2=-1 \quad \ldots(3)$$

$$2r_1+3=-3r_2+7 \text{ or } 2r_1+3r_2=4 \quad \ldots(4)$$

and

$$r_1-2=2r_2-7, \text{ or } r_1-2r_2=-5 \quad \ldots(5)$$

Solving (3) and (4), we get $r_1=-1$, $r_2=2$.

These values of r_1 and r_2 also satisfy the equation (5), and hence the given lines (1) and (2) intersect. Putting the value of r_1 (or of r_2), the point of intersection P (or Q) is $(2, 1, -3)$.

Now the equation of the plane in which the lines (1) and (2) lie is given by

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0$$

or $(x+1)(4+3)-(y-3)(-6-1)+(z+2)(9-2)=0$

or $$x+y+z=0.$$

Ex. 4. *In each of the following cases show that the two given lines are coplanar :*

(*i*) $\frac{1}{4}(x-5)=\frac{1}{4}(y-7)=-\frac{1}{5}(z+3)$; $\frac{1}{7}(x-8)=(y-4)=\frac{1}{3}(z-5)$.
Also find their point of intersection and the equation of the plane in which they lie.

(*ii*) $x=\frac{1}{2}(y-2)=\frac{1}{3}(z+3)$; $\frac{1}{2}(x-2)=\frac{1}{3}(y-6)=\frac{1}{4}(z-3)$.
Also find their point of intersection and the equation of the plane in which they lie. **[Indore 1979, Madras 75, 78]**

(*iii*) $\frac{1}{2}(x-1)=\frac{1}{3}(y-1)=\frac{1}{4}(z-1)$; $\frac{1}{3}(x-5)=\frac{1}{2}(y-7)=(z-9)$.
Also find the equation of the plane containing them. **[Ranchi 1976]**

(*iv*) $(x-1)=\frac{1}{2}(y-1)=\frac{1}{3}(z-1)$; $\frac{1}{2}(x-4)=\frac{1}{3}(y-6)=\frac{1}{3}(z-7)$.
Also find their point of intersection. **[Gorakhpur 1978]**

Sol. Proceed exactly as in Ex. 1 (2 or 3) above. The answers are

(i) $(1, 3, 2)$, $17x-47y-24z+172=0$.
(ii) $(2, 6, 3)$, $x-2y+z+7=0$.
(iii) $x-2y+z=0$. (iv) $(2, 5, 7)$.

Ex. 5. *Prove that the lines*

$$(x-a)/a'=(y-b)/b'=(z-c)/c'$$

and $$(x-a')/a=(y-b')/b=(z-c')/c$$

intersect and find the co-ordinates of the point of intersection and the equation of the plane in which they lie. **[Agra 1982]**

Sol. Any point on the first line is $P(a'r_1+a, b'r_1+b, c'r_1+c)$ and any point on the second line is $Q(ar_2+a', br_2+b', cr_2+c')$. The given lines will intersect if for some values of r_1 and r_2 the points P and Q coincide. Thus, we have

$$\left.\begin{aligned} a'r_1+a&=ar_2+a', \\ b'r_1+b&=br_2+b', \\ \text{and}\quad c'r_1+c&=cr_2+c' \end{aligned}\right\}.$$ These equations are clearly satisfied by $r_1=r_2=1$.

Hence the given lines intersect. Putting the value of r_1 (or of r_2), the point of intersection P (or Q) is $(a+a', b+b', c+c')$.

Now the equation of the plane in which the given lines lie is given by

$$\begin{vmatrix} x-a & y-b & z-c \\ a' & b' & c' \\ a & b & c \end{vmatrix}=0, \text{ or } \begin{vmatrix} x & y & z \\ a & b & c \\ a' & b' & c' \end{vmatrix}=0.$$

Ex. 6. *Show that the lines*

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a+d}{\alpha+\delta}$$

and $$\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar and find the equation of the plane in which they lie.

Sol. The given lines will be coplanar if

$$\begin{vmatrix} (b-c)-(a-d) & b-a & (b+c)-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0.$$ [See § 8 (A), condition (A)]

Adding the third column to the first column, the determinant on the left hand side

$$=\begin{vmatrix} 2(b-a) & b-a & b+c-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix}$$

$$=2\begin{vmatrix} b-a & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix}$$

$=0$, the first two columns being identical.

Hence the condition of coplanarity of the two lines is satisfied. Hence the given lines are coplanar.

Now the equation of the plane in which the given lines lie is given by

$$\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0.$$

Adding the third column to the first column, we get

$$\begin{vmatrix} x+z-2a & y-a & z-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix}=0.$$

Subtracting two times the second column from the first column, we get

$$\begin{vmatrix} x+z-2y & y-a & z-a-d \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix}=0$$

or $x+z-2y=0$ as the required equation.

Ex. 7. *Prove that the lines*

$$\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}, \frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}, \frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

will lie in one plane if

$$(l/\alpha)(b-c)+(m/\beta)(c-a)+(n/\gamma)(a-b)=0.$$

[Garhwal 1978; Kanpur 81, 83; Meerut 89]

Sol. We clearly see that the three given lines pass through the origin O and therefore they will be coplanar if they are perpendicular to a line through the origin O.

Let ξ, η, ζ be the d. c.'s of this line through the origin O. Hence if this line is perpendicular to the given lines, we have

$$\xi\alpha+\eta\beta+\zeta\gamma=0, \quad ...(1)$$

$$\xi a\alpha+\eta b\beta+\xi c\gamma=0, \quad ...(2)$$

and $$\xi l+\eta m+\zeta n=0. \quad ...(3)$$

Solving (1) and (2), we have

$$\frac{\xi}{c\beta\gamma-b\beta\gamma}=\frac{\eta}{a\alpha\gamma-c\alpha\gamma}=\frac{\zeta}{b\alpha\beta-a\alpha\beta}$$

or $$\frac{\xi}{(b-c)\beta\gamma}=\frac{\eta}{(c-a)\alpha\gamma}=\frac{\zeta}{(a-b)\alpha\beta}.$$

Putting these proportionate values of ξ, η, ζ in the equation (3), we have

$$l\beta\gamma(b-c)+m\alpha\gamma(c-a)+n\alpha\beta(a-b)=0.$$

Dividing throughout by $\alpha\beta\gamma$, we get

$$(l/\alpha)(b-c)+(m/\beta)(c-a)+(n/\gamma)(a-b)=0$$

as the required condition.

Ex. 8. *Show that the lines*

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}, \frac{x}{\alpha/a}=\frac{y}{\beta/b}=\frac{z}{\gamma/c}, \frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$$

are coplanar if $a=b$ or $b=c$ or $c=a$.

Sol. We clearly see that the three given lines pass through the origin O and therefore they will be coplanar if they are perpendicular to a line through the origin O. Let l, m, n be the d.c.'s of this line through the origin O. Hence if this line is perpendicular to the given lines, we have

$$la\alpha+mb\beta+nc\gamma=0. \quad ...(1)$$

$$l\alpha/a+m\beta/b+n\gamma/c=0, \quad ...(2)$$

and $$l\alpha+m\beta+n\gamma=0. \quad ...(3)$$

Eliminating $l\alpha$, $m\beta$ and $n\gamma$ between the equations (1), (2) and (3), we get the condition for the coplanarity of the given lines as

$$\begin{vmatrix} a & b & c \\ 1/a & 1/b & 1/c \\ 1 & 1 & 1 \end{vmatrix}=0.$$

Multiplying 1st, 2nd and 3rd columns by a, b, c respectively, we have

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}=0, \text{ or } \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=0.$$

Subtracting 1st column from 2nd and 3rd columns, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}=0, \text{ or } \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}=0,$$

or $$(b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}=0,$$

or $(b-a)(c-a)(c-b)=0$, as the required condition.

Hence the given lines are coplanar if

$$b=a \text{ or } c=a \text{ or } b=c.$$

Ex. 9. *Prove that the lines* $x=ay+b=cz+d$ *and* $x=\alpha y+\beta=\gamma z+\delta$ *are coplanar if* $(\alpha\beta-b\alpha)(\gamma-c)-(c\delta-d\gamma)(\alpha-a)=0$.

(Burdwan 1980; Kanpur 82)

Sol. The equations of the given lines are

$$x=ay+b=cz+d \text{ or } \frac{x-0}{1}=\frac{y+b/a}{1/a}=\frac{z+d/c}{1/c} \quad \ldots(1)$$

and $$x=\alpha y+\beta=\gamma z+\delta \text{ or } \frac{x-0}{1}=\frac{y+\beta/\alpha}{1/\alpha}=\frac{z+\delta/\gamma}{1/\gamma}. \quad \ldots(2)$$

The lines (1) and (2) will be coplanar if

$$\begin{vmatrix} 0-0 & \beta/\alpha-b/a & \delta/\gamma-d/c \\ 1 & 1/a & 1/c \\ 1 & 1/\alpha & 1/\gamma \end{vmatrix}=0,$$

[See § 8 (A), condition (A)]

or $$\begin{vmatrix} 0 & \beta/\alpha-b/a & \delta/\gamma-d/c \\ 0 & 1/a-1/\alpha & 1/c-1/\gamma \\ 1 & 1/\alpha & 1/\gamma \end{vmatrix}=0,$$

subtracting the third row from the second row

or $$\left(\frac{\beta}{\alpha}-\frac{b}{a}\right)\left(\frac{1}{c}-\frac{1}{\gamma}\right)-\left(\frac{\delta}{\gamma}-\frac{d}{c}\right)\left(\frac{1}{a}-\frac{1}{\alpha}\right)=0,$$

or $$\frac{(a\beta-b\alpha)(\gamma-c)}{\alpha a c\gamma}=\frac{(c\delta-d\gamma)(\alpha-a)}{c\gamma a\alpha}=0,$$

or $$(a\beta-b\alpha)(\gamma-c)-(c\delta-d\gamma)(\alpha-a)=0.$$

This is the required condition.

Ex. 10. *Prove that the lines*

$$3x-5=4y-9=3z \textit{ and } x-1=2y-4=3z$$

meet in a point and the equation of the plane in which they lie is

$$3x-8y+3z+13=0.$$ **(Kashmir 1975)**

Sol. The equations of the given lines are

$$3x-5=4y-9=3z \text{ or } \frac{x-5/3}{1/3}=\frac{y-9/4}{1/4}=\frac{z}{1/3}$$

or $$\frac{x-5/3}{4}=\frac{y-9/4}{3}=\frac{z}{4} \quad \ldots(1)$$

and $$x-1=2y-4=3z \text{ or } \frac{x-1}{1}=\frac{y-2}{1/2}=\frac{z}{1/3}$$

or $$\frac{x-1}{6}=\frac{y-2}{3}=\frac{z}{2}. \quad \ldots(2)$$

Now proceed as in Ex. 1 above.

Ex. 11. *Prove that the lines* $\frac{1}{2}(x-9)=-(y+4)=(z-5)$ *and* $6x+4y-5z=4,\ x-5y+2z=12$ *are coplanar. Find also their point of intersection and the equation of the plane in which they lie.*

(Agra 1980)

Sol. The equations of the given lines are

$$\frac{x-9}{2}=\frac{y+4}{-1}=\frac{z-5}{1}=r \text{ (say)} \quad \ldots(1)$$

and $$6x+4y-5z=4,\ x-5y+2z=12. \quad \ldots(2)$$

The equation of any plane through the line (2) is

$$(6x+4y-5z-4)+\lambda(x-5y+2z-12)=0$$

or $$(6+\lambda)x+(4-5\lambda)y+(-5+2\lambda)z-(4+12\lambda)=0. \quad \ldots(3)$$

Now if the plane (3) is parallel to the line (1), then we have

$$(6+\lambda)(2)+(4-5\lambda)(-1)+(-5+2\lambda)(1)=0, \text{ or } \lambda=-\tfrac{1}{3}.$$

Putting this value of λ in the equation (3), the equation of the plane through the line (2) and parallel to the line (1) is given by

$$(6-\tfrac{1}{3})x+(4+\tfrac{5}{3})y+(-5-\tfrac{2}{3})z-(4-12/3)=0$$

or $$17x+17y-17z=0, \text{ or } x+y-z=0. \quad \ldots(4)$$

Clearly the line (1) passes through the point $(9, -4, 5)$. This point $(9, -4, 5)$ satisfies the equation (4) of the plane. This shows that the line (1) lies in the plane (4). Hence both the given lines (1) and (2) are coplanar and lie in the plane given by (4).

To find the point of intersection.

The co-ordinates of any point on the line (1) are

$$(2r+9, -r-4, r+5). \quad \ldots(5)$$

The lines (1) and (2) intersect if this point also lies on the line (2) *i.e.*, if it satisfies both the equations of the line (2). Hence we have

$$6(2r+9)+4(-r-4)-5(r+5)=4, \text{ or } r=-3$$

and $$(2r+9)-5(-r-4)+2(r+5)=12, \text{ or } r=-3.$$

Since both the equations give the same value of r, the two given lines intersect. Putting the value of r in (5), the point of intersection is $(3, -1, 2)$.

Ex. 12. *Show that the lines* $\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$ *and* $4x-3y+1=0=5x-3z+2$ *are coplanar. Also find their point of intersection.* **(Burdwan 1978; Gorakhpur 79)**

Sol. The equations of the given lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r \text{ (say)}, \qquad \ldots(1)$$

and $$4x-3y+1=0,\ 5x-3z+2=0. \qquad \ldots(2)$$

The co-ordinates of any point on the line (1) are

$$(2r+1,\ 3r+2,\ 4r+3). \qquad \ldots(3)$$

The lines (1) and (2) will intersect *i.e.*, will be coplanar if this point also lies on the line (2). This point satisfies both the equations of the line (2), if we have

$$4\,(2r+1)-3\,(3r+2)+1=0, \text{ or } r=-1$$

and $$5\,(2r+1)-3\,(4r+3)+2=0, \text{ or } r=-1.$$

Since both the equations give the same value of r, the two given lines intersect. Putting this value of r in (3), the point of intersection is $(-1,\ -1,\ -1)$.

Ex. 13. *Show that the lines* $\frac{1}{3}(x+4)=\frac{1}{5}(y+6)=-\frac{1}{2}(z-1)$ *and* $3x-2y+z+5=0=2x+3y+4z-4$ *are coplanar. Also find their point of intersection and the equation of the plane in which they lie.*

Sol. The equations of the given lines are

$$\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}=r \text{ (say)} \qquad \ldots(1)$$

and $$3x-2y+z+5=0,\ 2x+3y+4z-4=0. \qquad \ldots(2)$$

The equation of any plane through the line (2) is

$$(3x-2y+z+5)+\lambda\,(2x+3y+4z-4)=0$$

or $$(3+2\lambda)\,x+(-2+3\lambda)\,y+(1+4\lambda)\,z+(5-4\lambda)=0. \qquad \ldots(3)$$

Now if the plane (3) is parallel to the line (1), then we have $(3+2\lambda)\,(3)+(-2+3\lambda)\,(5)+(1+4\lambda)\,(-2)=0$, or $\lambda=\frac{3}{13}$.

Putting this value of λ in the equation (3), the equation of the plane through the line (2) and parallel to the line (1) is given by

$$\left(3+\frac{6}{13}\right)x+\left(-2+\frac{9}{13}\right)y+\left(1+\frac{12}{13}\right)z+\left(5-\frac{12}{13}\right)=0$$

or $$45x-17y+25z+53=0. \qquad \ldots(4)$$

Clearly the line (1) passes through the point $(-4,\ -6,\ 1)$ and it satisfies the equation (4) as

$$45\,(-4)-17\,(-6)+25\,(1)+53=0.$$

Hence the lines (1) and (2) intersect and they lie in the plane (4).

To find the point of intersection.

The co-ordinates of any point on the line (1) are

$$(3r-4,\ 5r-6,\ -2r+1). \qquad \ldots(5)$$

The lines (1) and (2) intersect, if this point also lies on the line (2) *i.e.*, if it satisfies both the equations of the line (2).

Hence we have

$$3\,(3r-4)-2\,(5r-6)+(-2r+1)+5=0, \text{ or } r=2$$

and $$2\,(3r-4)+3\,(5r-6)+4\,(-2r+1)-4=0, \text{ or } r=2.$$

Since both the equations give the same value of r, the two given lines intersect. Putting the value of r in (5), the point of intersection is $(2,\ 4,\ -3)$.

Ex. 14. *Show that the lines*

$$7x-4y+7z+16=0=4x+3y-2z+3$$

and $$x-3y+4z+6=0=x-y+z+1$$

are coplanar. **(Gorakhpur 1975)**

Sol. If the given lines are to be coplanar, then we must have

$$\begin{vmatrix} 7 & -4 & 7 & 16 \\ 4 & 3 & -2 & 3 \\ 1 & -3 & 4 & 6 \\ 1 & -1 & 1 & 1 \end{vmatrix}=0. \qquad \text{[See § 8 (C)]}$$

Applying $R_1-7R_4,\ R_2-4R_4,\ R_3-R_4$, the determinant on the left hand side

$$=\begin{vmatrix} 0 & 3 & 0 & 9 \\ 0 & 7 & -6 & -1 \\ 0 & -2 & 3 & 5 \\ 1 & -1 & 1 & 1 \end{vmatrix}$$

$$=-\begin{vmatrix} 3 & 0 & 9 \\ 7 & -6 & -1 \\ -2 & 3 & 5 \end{vmatrix}, \text{ expanding the determinant along the first column}$$

$$=-\begin{vmatrix} 3 & 0 & 0 \\ 7 & -6 & -22 \\ -2 & 3 & 11 \end{vmatrix}, \text{ by } C_3-3C_1$$

$$=-3\,\{-66-(-66)\}=-3\,(-66+66)=0.$$

Hence the given lines are coplanar.

Ex. 15. *Show that the lines* $x+y+z-3=0=2x+3y+4z-5$

and $4x-y+5z-7=0=2x-5y-z-3$ *are coplanar, and find the plane in which they lie.* (Agra 1975)

Sol. We shall first transform the given equations to symmetrical forms. The equations of the given lines are

$$x+y+z-3=0=2x+3y+4z-5 \quad \text{...(1)}$$

and
$$4x-y+5z-7=0=2x-5y-z-3. \quad \text{...(2)}$$

Let l, m, n be the d.c.'s of the line (1); then we have

$$l+m+n=0, \quad \text{and} \quad 2l+3m+4n=0.$$

Solving, we get

$$\frac{l}{4-3}=\frac{m}{2-4}=\frac{n}{3-2} \quad \text{or} \quad \frac{l}{1}=\frac{m}{-2}=\frac{n}{1}.$$

$\therefore$ the d.c.'s of the line (1) are proportional to $1, -2, 1$.

Now let us find the point where the line (1) meets the plane $z=0$. The co-ordinates of this point are given by

$$x+y-3=0,\ 2x+3y-5=0.$$

Solving, $x=4,\ y=-1$.

Hence the line (1) passes through the point $(4, -1, 0)$. Thus the symmetrical form of the line (1) is

$$\frac{x-4}{1}=\frac{y+1}{-2}=\frac{z-0}{1}=r_1 \text{ (say)}. \quad \text{...(3)}$$

Again let l_1, m_1, n_1 be the d.c.'s of the line (2); then we have

$$4l_1-m_1+5n_1=0, \quad \text{and} \quad 2l_1-5m_1-n_1=0.$$

Solving, $\dfrac{l_1}{1+25}=\dfrac{m_1}{10+4}=\dfrac{n_1}{-20+2}$ or $\dfrac{l_1}{13}=\dfrac{m_1}{7}=\dfrac{n_1}{-9}$.

$\therefore$ the d.c.'s of the line (2) are proportional to $13, 7, -9$.

Now let the line (2) meet the plane $z=0$. The co-ordinates of this point are given by

$$4x-y-7=0,\ 2x-5y-3=0.$$

Solving, $x=16/9,\ y=1/9$.

Hence the line (2) passes through the point $(16/9, 1/9, 0)$.

Thus the symmetrical form of the line (2) is

$$\frac{x-\frac{16}{9}}{13}=\frac{y-\frac{1}{9}}{7}=\frac{z-0}{-9}=r_2 \text{ (say)}. \quad \text{...(4)}$$

The co-ordinates of any point P on the line (3) are

$$(r_1+4,\ -2r_1-1,\ r_1)$$

and that of any point Q on the line (4) are

$$(13r_2+16/9,\ 7r_2+1/9,\ -2r_2).$$

If the lines (3) and (4) [*i.e.* (1) and (2)] are coplanar, then for some values of r_1 and r_2 the points P and Q coincide. Thus we have

$$r_1+4=13r_2+(16/9). \quad \text{...(5)}$$
$$-2r_1-1=7r_2+(1/9), \quad \text{...(6)}$$
and
$$r_1=-9r_2. \quad \text{...(7)}$$

Solving (5) and (7), we get $r_1=-10/11,\ r_2=10/99$.

The values of r_1 and r_2 also satisfy the equation (6) and hence the given lines are coplanar. Putting the value of r_1 (or of r_2) the **point of intersection** P (or Q) is $(-9/11, -8/11, -7/11)$.

Now the equation of the plane in which the lines (3) and (4) [*i.e.* (1) and (2)] lie is given by

$$\begin{vmatrix} x-4 & y+1 & z \\ 1 & -2 & 1 \\ 13 & 7 & -9 \end{vmatrix}=0 \quad [\text{See § 8 (A)}]$$

or $(x-4)\{(-2)(-9)-7.1\}-(y+1)\{1.(-9)-1(13)\}$
$+z\{1.7-(-2)(13)\}=0$

or $(x-4).11+(y+1).22+z.(33)=0$ or $x+2y+3z=2$.

Ex. 16. *A, A'; B, B'; C, C' are points on the axes. Show that the lines of intersection of the planes $A'BC$, ABC'; $B'CA$, $BC'A'$; $C'AB$, $CA'B'$ are coplanar.*

Sol. Let the co-ordinates of the points A and A' be $(a, 0, 0)$ and $(a', 0, 0)$; B and B' be $(0, b, 0)$ and $(0, b', 0)$; C and C' be $(0, 0, c)$ and $(0, 0, c')$.

The equations (in the intercepts form) of the planes $A'BC$ and $AB'C'$ are respectively given by

$$\frac{x}{a'}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{and} \quad \frac{x}{a}+\frac{y}{b'}+\frac{z}{c'}=1.$$

Therefore, the equation of any plane through the line of intersection of these two planes is given by

$$\left(\frac{x}{a'}+\frac{y}{b}+\frac{z}{c}-1\right)+\lambda\left(\frac{x}{a}+\frac{y}{b'}+\frac{z}{c'}-1\right)=0, \quad \text{...(1)}$$

for some value of the constant λ.

Choosing $\lambda=1$, the equation (1) of the plane [containing the line of intersection of the planes $A'BC$ and $AB'C'$] becomes

$$\left(\frac{x}{a'}+\frac{y}{b}+\frac{z}{c}-1\right)+\left(\frac{x}{a}+\frac{y}{b'}+\frac{z}{c'}-1\right)=0$$

or
$$\left(\frac{1}{a'}+\frac{1}{a}\right)x+\left(\frac{1}{b}+\frac{1}{b'}\right)y+\left(\frac{1}{c}+\frac{1}{c'}\right)z=2. \quad \text{...(2)}$$

The symmetry in the equation (2) shows that the lines of intersection of the other two pairs of planes also lie in the plane given by (2).

Hence the lines of intersection of the given three pairs of planes are coplanar.

Ex. 17. *Find the equation of the plane through the line*

$$x/l=y/m=z/n$$

and perpendicular to the plane containing the lines

$$x/m=y/n=z/l \quad \text{and} \quad x/n=y/l=z/m.$$

(Lucknow 1982, Ranchi 79, Punjab 77, Agra 75, M.U. 89(S))

Sol. The equation of any plane through the line

$x/l=y/m=z/n$ is

$$Ax+By+Cz=0, \quad \ldots(1)$$

where $$Al+Bm+Cn=0. \quad \ldots(2)$$

Also the equation of the plane through the lines

$$x/m=y/n=z/l \quad \text{and} \quad x/n=y/l=z/m$$

(note that both of these lines are passing through the origin) is

$$\begin{vmatrix} x & y & z \\ m & n & l \\ n & l & m \end{vmatrix}=0, \text{ [See § 8(A)]}$$

or $$(mn-l^2)\,x+(ln-m^2)\,y+(ml-n^2)\,z=0. \quad \ldots(3)$$

According to the question, the planes (1) and (3) should be perpendicular the condition for which is

$$A\,(mn-l^2)+B\,(ln-m^2)+C\,(ml-n^2)=0. \quad \ldots(4)$$

Solving the relations (2) and (4), we get

$$\frac{A}{m\,(ml-n^2)-n\,(ln-m^2)}=\frac{B}{n\,(mn-l^2)-l\,(ml-n^2)}=\frac{C}{l\,(ln-m^2)-m\,(mn-l^2)}$$

or $$\frac{A}{m^2l-mn^2-n^2l+nm^2}=\frac{B}{mn^2-nl^2-ml^2+ln^2}=\frac{C}{l^2n-lm^2-m^2n+ml^2}$$

or $$\frac{A}{(m^2-n^2)\,l+mn\,(m-n)}=\frac{B}{(n^2-l^2)\,m+nl\,(n-l)}=\frac{C}{(l^2-m^2)\,n+lm\,(l-m)}$$

or $$\frac{A}{(m-n)\,(ml+nl+mn)}=\frac{B}{(n-l)\,(nm+lm+nl)}$$

$$=\frac{C}{(l-m)\,(ln+mn+lm)}$$

or $$\frac{A}{m-n}=\frac{B}{n-l}=\frac{C}{l-m}$$

Putting these proportionate values of A, B, C in (1), the equation of the required plane is given by

$$(m-n)+(n-l)\,y+(l-m)\,z=0.$$

Ex. 18. *Find the foot and hence the length of the perpendicular from the point* (5, 7, 3) *to the line*

$$(x-15)/3=(y-29)/8=(z-5)/(-5).$$

Find the equations of the perpendicular. Also find the equation of the plane in which the perpendicular and the given straight line lie.

Sol. Let the given point (5, 7, 3) be P.

The equations of the given line are

$$(x-15)/3=(y-29)/8=(z-5)/(-5)=r \text{ (say)}. \quad \ldots(1)$$

Let N be the foot of the perpendicular from the point P to the line (1). The co-ordinates of N may be taken as

$$(3r+15,\ 8r+29,\ -5r+5). \quad \ldots(2)$$

∴ the direction ratios of the perpendicular PN are

$$3r+15-5,\ 8r+29-7,\ -5r+5-3,$$

i.e. are $3r+10,\ 8r+22,\ -5r+2.$ ...(3)

Since the line (1) and the line PN are perpendicular to each other, therefore $3\,(3r+10)+8\,(8r+22)-5\,(-5r+2)=0$

or $$98r+196=0 \quad \text{or} \quad r=-2.$$

Putting this value of r in (2) and (3), the foot of the perpendicular N is (9, 13, 15) and the direction ratios of the perpendicular PN are 4, 6, 12 or 2, 3, 6.

∴ the equations of the perpendicular PN are

$$(x-5)/2=(y-7)/3=(z-3)/6. \quad \ldots(4)$$

Length of the perpendicular PN

= the distance between $P(5, 7, 3)$ and $N(9, 13, 15)$

$=\sqrt{\{(9-5)^2+(13-7)^2+(15-3)^2\}}=14.$

Lastly the equation of the plane containing the given line (1) and the perpendicular (4) is given by

$$\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix}=0 \text{ [See § 8 (A)]}$$

or $(x-15)(48+15)-(y-29)(18+10)+(z-5)(9-16)=0$

or $9x-4y-z-14=0.$

§ 9. To determine the equations of a straight line intersecting two given lines.

Case I. *If the equations of the two lines are given in symmetrical forms.*

Let the equations of the given lines be

$$(x-\alpha_1)/l_1=(y-\beta_1)/m_1=(z-\gamma_1)/n_1=r_1 \text{ (say)}, \quad ...(1)$$

and $$(x-\alpha_2)/l_2=(y-\beta_2)/m_2=(z-\gamma_2)/n_2=r_2 \text{ (say)}. \quad ...(2)$$

The co-ordinates of any point P on (1) are

$$(l_1r_1+\alpha_1,\ m_1r_1+\beta_1,\ n_1r_1+\gamma_1)$$

and those of any point Q on (2) are

$$(l_2r_2+\alpha_2,\ m_2r_2+\beta_2,\ n_2r_2+\gamma_2).$$

Now we are to find the equations of a line which intersects the lines (1) and (2). Suppose the required line intersects the lines (1) and (2) in the points P and Q respectively.

Then the required line is one which joins the points P and Q. The values of r_1 and r_2 will be determined by some additional given conditions.

Case II. *If the equations of the two given line are given in general form.*

Let the equations of the given lines be

$$u_1=0=v_1 \text{ and } u_2=0=v_2.$$

Now the equations of the required line intersecting both the given lines are $u_1+\mu_1v_1=0$ and $u_2+\mu_2v_2=0$,

where the values of μ_1 and μ_2 are determined by some additional given conditions.

SOLVED EXAMPLES (G)

Ex 1. *A line with direction cosines proportional to* (2, 7, −5) *is drawn to intersect the lines*

$$(x-5)/3=(y-7)/(-1)=(z+2)/1$$

and $$(x+3)/(-3)=(y-3)/2=(z-6)/4.$$

Find the co-ordinates of the points of intersection and the length intercepted on it. **[Kanpur 1980, Lucknow 79, Bundelkhand 78]**

Sol. The equations of the given lines are

$$(x-5)/3=(y-7)/(-1)=(z+2)/1=r_1 \text{ (say)}, \quad ...(1)$$

and $$(x+3)/(-3)=(y-3)/2=(z-6)/4=r_2 \text{ (say)}. \quad ...(2)$$

Any point P on (1) is $(3r_1+5, -r_1+7, r_1-2)$,

and any point Q on (2) is $(-3r_2-3, 2r_2+3, 4r_2+6)$.

The direction ratios of PQ are

$$(3r_1+3r_2+8,\ -r_1-2r_2+4,\ r_1-4r_2-8). \quad ...(3)$$

Suppose the line with d.r.'s (2, 7, −5) meets the lines (1) and (2) in the points P and Q respectively.

Then the d.r.'s 2, 7, −5 will be proportional to the d.r.'s given by (3).

$$\therefore \quad \frac{3r_1+3r_2+8}{2}=\frac{-r_1-2r_2+4}{7}=\frac{r_1-4r_2-8}{-5}. \quad ...(4)$$

From the first two of (4), we get

$$7(3r_1+3r_2+8)=2(-r_1-2r_2+4)$$

or $$23r_1+25r_2+48=0. \quad ...(5)$$

And from the 1st and 3rd of (4), we get

$$2(r_1-4r_2-8)=-5(3r_1+3r_2+8)$$

or $$17r_1+7r_2+24=0. \quad ...(6)$$

Solving (5) and (6), we have $r_1=r_2=-1$.

Now putting these values of r_1 and r_2 the co-ordinates of the points of intersection are $P(2, 8, -3)$ and $Q(0, 1, 2)$.

The required length intercepted by the lines (1) and (2) on the line with d.r.'s (2, 7, −5)

$$=PQ=\sqrt{[(2-0)^2+(8-1)^2+(-3-2)^2]}$$
$$=\sqrt{(4+49+25)}=\sqrt{(78)}.$$

Note. The equations of the line PQ in the above **Ex. 1** are given by $(x-2)/2=(y-8)/7=(z+3)/(-5)$.

Ex. 2. *Find the equations to the planes through the point* (1, 0, −1) *and the lines*

$$4x-y-13=0=3y-4z-1 \textit{ and } y-2z+2=0=x-5$$

and show that the equations to the line through the given point which intersects the two given lines can be written as $x=y+1=z+2$.

Sol. The equations of the given lines are

$$4x-y-13=0,\ 3y-4z-1=0 \quad ...(1)$$

and $$y-2z+2=0,\ x-5=0. \quad ...(2)$$

The equations of any planes through the given lines (1) and (2) are respectively given by

$$4x-y-13+\mu_1(3y-4z-1)=0,$$

and $$(y-2z+2)+\mu_2(x-5)=0.$$

If these planes pass through the point (1, 0, −1), we get $4-0-13+\mu_1(0+4-1)=0$ giving $\mu_1=3$, and $0+2+2+\mu_2(1-5)=0$ giving $\mu_2=1$.

Putting these values of μ_1 and μ_2, the required equations of the planes are

$$4x-y-13+3(3y-4z-1)=0,$$

and $(y-2z+2)+1.(x-5)=0$

or $x+2y-3z-4=0$, and $x+y-2z-3=0$. ...(3)

The equations (3) are the general equations of a line through the given point (1, 0, −1) and intersecting the given lines (1) and (2).

Transforming the equations (3) to the symmetrical form we get

$$\frac{x-0}{1}=\frac{y+1}{1}=\frac{z+2}{1} \text{ or } x=y+1=z+2.$$

Ex. 3. *Find the equations to the straight line drawn from the origin to intersect the lines*

$$2x+5y+3z-4=0=x-y-5z-6$$

and $3x-y+2z-1=0=x+2y-z-2.$

Sol. The equations of any line intersecting the given lines are

$(2x+5y+3z-4)+\mu_1(x-y-5z-6)=0,$

and $(3x-y+2z-1)+\mu_2(x+2y-z-2)=0.$ [See § 9 (II)]

If this line passes through (0, 0, 0), then we have

$-4-6\mu_1=0$ or $\mu_1=-2/3$; and $-1-2\mu_2=0$ or $\mu_2=-1/2$.

Putting these values of μ_1 and μ_2, the equations of the required line are

$(2x+5y+3z-4)-(2/3)(x-y-5z-6)=0$

and $(3x-y+2z-1)-\frac{1}{2}(x+2y-z-2)=0$

or $4x+17y+19z=0$ and $5x-4y+5z=0$.

Ex. 4. *Find the equations to the line drawn parallel to* $\frac{1}{4}x=y=z$ *so as to meet the lines* $5x-6=4y+3=z$ *and*

$$2x-4=3y+5=z.$$

Sol. The general equations of the line $5x-6=4y+3=z$ may be written as

$5x-6-(4y+3)=0$ and $5x-6-z=0$

or $5x-4y-9=0=5x-z-6$. ...(1)

Similarly the general equations of the line $2x-4=3y+5=z$ may be written as

$$2x-3y-9=0=2x-z-4. \quad ...(2)$$

The equations of any line intersecting the given lines [i.e. the lines (1) and (2)] are

$(5x-4y-9)+\mu_1(5x-z-6)=0$

and $(2x-3y-9)+\mu_2(2x-z-4)=0$

or $5(1+\mu_1)x-4y-\mu_1 z-(9+6\mu_1)=0$

and $2(1+\mu_2)x-3y-\mu_2 z-(9+4\mu_2)=0$ ...(3)

If the line (3) is parallel to the line $\frac{x}{4}=\frac{y}{1}=\frac{z}{1}$, then this latter line is perpendicular to the normals of each of the two planes given by (3), so that we have

$4\cdot5(1+\mu_1)+1.(-4)+1.(-\mu_1)=0$ giving $\mu_1=-16/19$

and $4\cdot2(1+\mu_2)+1.(-3)+1.(-\mu_2)=0$ giving $\mu_2=-5/7$.

Putting these values of μ_1 and μ_2 in (3) and simplifying, the required equations of the line are

$$15x-76y+16z-75=0 \text{ and } 4x-21y+5z-43=0.$$

Ex. 5. *A line with direction cosines proportional to 2, 1, 2 meets each of the lines given by the equations*

$$x=y+a=z,\ x+a=2y=2z.$$

Find the co-ordinates of each of the points of intersection. **(Utkal 1977 S)**

Sol. The equations of the given lines are

$x/1=(y+a)/1=z/1=r_1$ (say), ...(1)

and $(x+a)/2=y/1=z/1=r_2$ (say). ...(2)

Any point P on (1) is (r_1, r_1-a, r_1),

and any point Q on (2) is $(2r_2-a, r_2, r_2)$.

The d.r.'s of PQ are $r_1-2r_2+a,\ r_2-r_1-a,\ r_1-r_2$.

If the d.r.'s of PQ are proportional to 2, 1, 2, then

$$\frac{r_1-2r_2+a}{2}=\frac{r_1-r_2-a}{1}=\frac{r_1-r_2}{2}. \quad ...(3)$$

From the first two of (3), we get $r_1-3a=0$ or $r_1=3a$ and from the last two of (3), we get $r_1-r_2-2a=0$ or $r_2=a$.

Putting these values of r_1 and r_2, the co-ordinates of the points of intersection are $P(3a, 2a, 3a)$ and $Q(a, a, a)$.

Ex. 6. *Find the equations to the line intersecting the lines*

$$x-1=y-z-1,\ 2x+2=2y=z+1$$

and parallel to the line $\frac{1}{2}(x-1)=(y-1)=\frac{1}{3}(z-2)$. **(Agra 1977)**

Sol. Here the required line being parallel to the given line $\frac{1}{2}(x-1)=(y-1)=\frac{1}{3}(z-2)$, will have its d.r.'s 2, 1, 3.

Proceeding as in Ex. 1 above, the equations of the required line PQ are $\frac{1}{2}(x-1)=y=\frac{1}{3}(z-1)$.

Ex. 7. *Find the equations of the line intersecting the lines* $x-a=y=z-a,\ x+a=y=\frac{1}{2}(z+a)$ *and parallel to the line* $\frac{1}{2}(x-a)=(y-a)=\frac{1}{3}(z-2a)$.

Sol. The equations of the given lines are

$(x-a)/1=y/1=(z-a)/1=r_1$ (say) ...(1)

and $(x+a)/1=y/1=(z+a)/2=r_2$ (say). ...(2)

Any point on (1) is $P(r_1+a, r_1, r_1+a)$ and any point on (2) is $Q(r_2-a, r_2, 2r_2-a)$.

The d.r.'s of PQ are r_1-r_2+2a, r_1-r_2, r_1-2r_2+2a.

If the line PQ is parallel to the given line $\frac{x-a}{2}=\frac{y-a}{1}=\frac{z-2a}{3}$, then the d.r.'s of PQ are proportional to 2, 1, 3.

$$\therefore \frac{r_1-r_2+2a}{2}=\frac{r_1-r_2}{1}=\frac{r_1-2r_2+2a}{3}. \quad ...(3)$$

From the first two of (3), we get $r_1-r_2=2a$
and from the last two of (3), we get $2r_1-r_2=2a$ } ...(4)

Solving the equations (4), we get $r_1=0$, $r_2=-2a$.

Putting these values of r_1 and r_2, the co-ordinates of the points of intersection are $P(a, 0, a)$ and $Q(-3a, -2a, -5a)$.

$\therefore$ the equations of PQ are $(x-a)/2=y/1=(z-a)/3$.

Ex. 8. *Find the equations of the straight line through the origin and cutting each of the lines*

$$(x-x_1)/l_1=(y-y_1)/m_1=(z-z_1)/n_1$$

and $$(x-x_2)/l_2=(y-y_2)/m_2=(z-z_2)/n_2.$$

Sol. The equation of a plane through the first line namely $(x-x_1)/l_1=(y-y_1)/m_1=(z-z_1)/n_1$ is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0, \quad ...(1)$$

where $$Al_1+Bm_1+Cn_1=0. \quad ...(2)$$

If the plane (1) passes through (0, 0, 0), then

$$Ax_1+By_1+Cz_1=0. \quad ...(3)$$

Eliminating A, B, C from (1), (3) and (2), the equation of the plane through the origin and through the first line is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_1 & y_1 & z_1 \\ l_1 & m_1 & n_1 \end{vmatrix}=0.$$

Adding the second row to the first row, we get

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ l_1 & m_1 & n_1 \end{vmatrix}=0 \quad ...(4)$$

or $$(n_1y_1-m_1z_1)x+(l_1z_1-n_1x_1)y+(m_1x_1-l_1y_1)z=0. \quad ...(4')$$

Similarly the plane through the origin and through the second line is

$$\begin{vmatrix} x & y & z \\ x_2 & y_2 & z_2 \\ l_2 & m_2 & n_2 \end{vmatrix}=0. \quad ...(5)$$

or $$(n_2y_2-m_2z_2)x+(l_2z_2-n_2x_2)y+(m_2x_2-l_2y_2)z=0 \quad (5')$$

The planes (4) [or (4')] and 5 [or (5') together give the required line.

Ex 9. *Find the equations to the straight line drawn through the origin which will intersect both the lines*

$$\frac{x-1}{1}=\frac{y+3}{4}=\frac{z-5}{3} \text{ and } \frac{x-4}{2}=\frac{y+3}{3}=\frac{z-14}{4}.$$

Sol. The equation of any plane through the first line is

$$A(x-1)+B(y+3)+C(z-5)=0, \quad ...(1)$$

where $$1.A+4.B+3.C=0. \quad ...(2)$$

If the plane (1) passes through (0, 0, 0), we have

$$-A+3B-5C=0. \quad ...(3)$$

Solving (2) and (3), we get $A/29=B/(-2)=C/(-7)$.

Substituting these proportionate values of A, B, C in (1), the equation of the plane through the origin and the first line is

$$29x-2y-7z=0. \quad ...(4)$$

Similarly the equation of the plane through the origin and the second line is

$$9x-2y-3z=0. \quad ..(5)$$

The planes (4) and (5) together give the required line.

Ex. 10. *Find the equations of the line through the point* (3, 1, 2) *parallel to the plane* $4x+y+5z=0$ *so as to cut the line* $x+3=y+1=2(z-2)$. *Find also the point of intersection.*

Sol. Let the point P be (3, 1, 2).

The equation of the given plane is $4x+y+5z=0$. ..(1)

The equations of the given line may be written as

$$(x+3)/2=(y+1)/2=(z-2)/1=r \text{ (say)}. \quad ...(2)$$

Let the line through P (3, 1, 2) parallel to the plane (1) cut the line (2) in the point Q.

The point Q may be taken as $(2r-3, 2r-1, r+2)$.

$\therefore$ the d.r.'s of PQ are $2r-6$, $2r-2$, r. ..(3)

But PQ is parallel to the plane (1) and hence PQ is perpendicular to the normal of the plane (1) whose d.r.'s are 4, 1, 5 and so we have

$$(2r-6)\,4+(2r-2).1+r.5=0 \text{ or } r=26/15.$$

Putting this value of r, the co-ordinates of the point of intersection Q are (7/15, 37/15, 56/15).

Also from (3), the d.r.'s of PQ are

$$-38/15,\ 22/16,\ 26/15 \quad \text{or} \quad 19,\ -11,\ -13.$$

$\therefore$ the equations of the line PQ are

$$\frac{x-3}{19}=\frac{y-1}{-11}=\frac{z-2}{-13}.$$

Ex. 11. *Find the equations of the line through* (a, b, c) *which is parallel to the plane* $lx+my+nz=0$ *and intersects the line*

$$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2.$$

Sol. The required line is the line of intersection of the two planes given below.

(i) The plane passing through the point (a, b, c) and parallel to the plane $lx+my+nz=0$. The equation of this plane is clearly given by

$$l\,(x-a)+m\,(y-b)+n\,(z-c)=0. \qquad \ldots(1)$$

(ii) The plane through the point (a, b, c) and the given line

$$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2. \qquad \ldots(2)$$

The equation of any plane through the line (2) is

$$(a_1x+b_1y+c_1z+d)+\mu\,(a_2x+b_2y+c_2z+d_2)=0. \qquad \ldots(3)$$

If this plane (3) passes through (a, b, c), we get

$$(a_1a+b_1b+c_1c+d_1)+\mu\,(a_2a+b_2b+c_2c+d_2)=0$$

or

$$\mu=-(a_1a+b_1b+c_1c+d_1)/(a_2a+b_2b+c_2c+d_2).$$

Putting this value of μ in (3), the equation of the required second plane is given by

$$\frac{a_1x+b_1y+c_1z+d_1}{a_1a+b_1b+c_1c+d_1}=\frac{a_2x+b_2y+c_2z+d_2}{a_2a+b_2b+c_2c+d_2} \qquad \ldots(4)$$

Hence the equations of the required line are given by the planes (1) and (4).

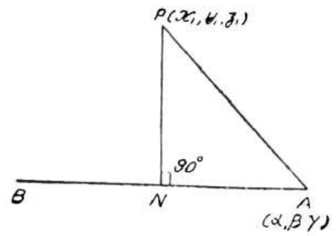
§ 10. To find the perpendicular distance of a point from a line and the co-ordinates of the foot of the perpendicular.

(Punjab 1979; Meerut 84S)

Let $P\,(x_1, y_1, z_1)$ be a given point and let AB be a given line. Let the equations of AB in symmetrical form be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \qquad \ldots(1)$$

where l, m, n are the d.c.'s of (1). The line (1) is passing through the point $A\,(\alpha, \beta, \gamma)$ and has direction cosines l, m, n. From P draw PN perpendicular to AB. Now it is required to find PN. From the right angled $\triangle APN$, we have



$$PN^2=AP^2-AN^2. \qquad \ldots(2)$$

Now AP=the distance between (α, β, γ) and $P\,(x_1, y_1, z_1)$

$$=\sqrt{\{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2\}}, \qquad \ldots(5)$$

and AN=projection of AP on AB *i.e.* the projection of AP on a line whose d.c.'s are l, m, n

$$=(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n. \qquad \ldots(4)$$

Putting the values from (3) and (4) in (2), we get

$$PN^2=\{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2\}-\{(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n\}^2$$

$$=\{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2\}\,(l^2+m^2+n^2)-\{(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n\}^2$$

$$[\because\ l^2+m^2+n^2=1]$$

$$=\{m\,(z_1-\gamma)-n\,(y_1-\beta)\}^2+\{n\,(x_1-\alpha)-l\,(z_1-\gamma)\}^2+\{l\,(y_1-\beta)-m\,(x_1-\alpha)\}^2$$

[By using Lagrange's identity]

$$=\begin{vmatrix} m & n \\ y_1-\alpha & z_1-\gamma \end{vmatrix}^2+\begin{vmatrix} n & l \\ z_1-\gamma & x_1-\alpha \end{vmatrix}^2+\begin{vmatrix} l & m \\ x_1-\alpha & y_1-\beta \end{vmatrix}^2. \qquad \ldots(5)$$

Note. In the equations (1) of the line AB, l, m, n have been taken as the actual direction cosines of the line. In case direction ratios a, b, c of AB are given, we should either first find the direction cosines of AB or we should divide the R.H.S. of (5) by $(a^2+b^2+c^2)$.

To find the co-ordinates of the foot of the perpendicular N.

Since N, the foot of the perpendicular, is a point on the line AB given by (1), its co-ordinates may be written as

$$(lr+\alpha,\ mr+\beta,\ nr+\gamma). \qquad \ldots(6)$$

The d.r.'s of PN are $lr+\alpha-x_1$, $mr+\beta-y_1$, $nr+\gamma-z_1$.

Also PN is perpendicular to AB.

$\therefore \quad (lr+\alpha-x_1)\, l+(mr+\beta-y_1).m+(nr+\gamma-z_1).n=0$

or $\quad r\,(l^2+m^2+n^2)=l\,(x_1-\alpha)+m\,(y_1-\beta)+n\,(z_1-\gamma)$

or $\quad r=l\,(x_1-\alpha)+m\,(y_1-\beta)+n\,(z_1-\gamma) \qquad [\because\ l^2+m^2+n^2=1]$

Putting this value of r in (6) the co-ordinates of N are obtained.

SOLVED EXAMPLES (H)

Ex. 1. *From the point P (1, 2, 3) PN is drawn perpendicular to the straight line $\frac{1}{3}(x-2)=\frac{1}{4}(y-3)=\frac{1}{5}(z-4)$. Find the distance PN, the equations to PN and co-ordinates of N.* **(Rohilkhand 198)**

Sol. The equations of the given line AB (say) are

$$(x-2)/3=(y-3)/4=(z-4)/5=r \text{ (say)}. \qquad ...(1)$$

The line (1) is passing through the point A (2, 3, 4). Since N, the foot of the perpendicular, is a point on the line (1) [*i.e.* AB] the co-ordinates of N may be written as

$$(3r+2,\ 4r+3,\ 5r+4) \qquad ...(2)$$

$\therefore$ the d.r.'s of PN are $3r+2-1,\ 4r+3-2,\ 5r+4-3$ *i.e.* are $3r+1,\ 4r+1,\ 5r+1$. ...(3)

The d.r.'s of the line AB whose equations are given by (1), are 3, 4, 5.

Since PN is perpendicular to AB, we have

$$3.(3r+1)+4.(4r+1)+5.(5r+1)=0, \quad \text{or} \quad r=-6/25.$$

Putting the value of r in (2), we get

$$N\equiv(32/25,\ 51/25,\ 14/5).$$

$\therefore \quad PN=$ the distance between the points P and N

$$=\sqrt{\left\{\left(\frac{32}{25}-1\right)^2+\left(\frac{51}{25}-2\right)^2+\left(\frac{14}{5}-3\right)^2\right\}}=\frac{\sqrt{3}}{5}.$$

Putting the value of r in (3), the d.r.'s of PN are 7/25, 1/25, $-5/25$ *i.e.* are 7, 1, -5.

$\therefore$ the equations to PN *i.e.* of a line passing through P (1, 2, 3) and having d.r.'s 7, 1, -5 are

$$\frac{x-1}{7}=\frac{y-2}{1}=\frac{z-3}{-5}.$$

Ex. 2. *How far is the point (4, 1, 1) from the line of intersection of $x+y+z-4=0=x-2y-z-4$?*

Sol. Let the point (4, 1, 1) be taken as P.

The equations of the given line AB (say) are

$$(x-4)+y+z=0,\ (x-4)-2y-z=0.$$

Solving for $x-4,\ y,\ z$, we get

$$\frac{x-4}{-1+2}=\frac{y}{1+1}=\frac{z}{-2-1} \quad \text{or} \quad \frac{x-4}{1}=\frac{y}{2}=\frac{z}{-3}. \qquad ...(1)$$

Equations (1) are the symmetrical form of the given line AB. The line (1) has d.r.'s 1, 2, -3 and passes through the point A (4, 0, 0).

Draw PN perpendicular to the line AB given by (1), so that N, the foot of perpendicular, lies on (1). Hence the co-ordinates of N may be written as $\quad (r+4,\ 2r,\ -3r)$. ...(2)

The d.r.'s of PN are

$$r+4-4,\ 2r-1,\ -3r-1 \text{ i.e. } r,\ 2r-1,\ -3r-1.$$

Since PN is perpendicular to AB, we have

$$r.1+(2r-1).2+(-3r-1)(-3)=0, \quad \text{or } r=-1/14.$$

Putting the value of r in (2), we get

$$N\equiv(55/14,\ -1/7,\ 3/14)$$

$\therefore$ the distance of P (4, 1, 1) from the given line (1)

= the distance between the points P and N

$$=\sqrt{\left\{\left(\frac{55}{14}-4\right)^2+\left(-\frac{1}{7}-1\right)^2+\left(\frac{3}{14}-1\right)^2\right\}}=\frac{\sqrt{(378)}}{14}=\frac{3\sqrt{(42)}}{14}.$$

Ex. 3. *Find the equations of the perpendicular from* (1, 3, 7) *on the line $x=3-5t,\ y=2+5t,\ z=-7+2t$.* **(Punjab 1981)**

Sol. The equations of the given line, say AB, may be written as

$$(x-3)/(-5)=(y-2)/5=(z+7)/2=t. \qquad ...(1)$$

Now proceed as in Ex. 1 above and get

$$N\equiv(-53/54,\ 323/54,\ -212/54)$$

and the equations to PN are

$$\frac{x-1}{107}=\frac{y-3}{-161}=\frac{z-7}{670}$$

Ex. 4. *Find the locus of a point which moves so that its distance from the line $x=y=-z$ is twice its distance from the plane $x-y+z=1$.* **(Agra 80)**

Sol. Let the given point be $P\,(x_1, y_1, z_1)$ whose locus is required to be found.

The equations of the given line AB are

$$x/1=y/1=z/(-1). \qquad ...(1)$$

The line (1) is clearly passing through A (0, 0, 0) and has d.r.'s 1, 1, -1. Hence d.c.'s $l,\ m,\ n$ of (1) are $1/\sqrt{3},\ 1/\sqrt{3}$, $-1/\sqrt{3}$. Let p_1 be the perpendicular distance of P from the line (1), then by § 10, we have

$$p_1^2=\begin{vmatrix} 1/\sqrt{3} & -1/\sqrt{3} \\ y_1-0 & z_1-0 \end{vmatrix}^2+\begin{vmatrix} -1/\sqrt{3} & 1/\sqrt{3} \\ z_1-0 & x_1-0 \end{vmatrix}^2+\begin{vmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ x_1-0 & y_1-0 \end{vmatrix}^2$$

$$p_1^2=\tfrac{1}{3}\{(z_1+y_1)^2+(-x_1-z_1)^2+(y_1-x_1)^2\}$$

or $$=\tfrac{2}{3}(x_1^2+y_1^2+z_1^2+y_1z_1+z_1x_1-x_1y_1) \quad ...(2)$$

Let p_2 be the perpendicular distance of $P(x_1, y_1, z_1)$ from the plane $x-y+z=1$. Then

$$p_2=\frac{x_1-y_1+z_1-1}{\sqrt{(1+1+1)}}. \quad ...(3)$$

According to the given problem, $p_1=2p_2$.

Squaring, $\quad p_1^2=4p_2^2$

or $\quad \tfrac{2}{3}(x_1^2+y_1^2+z_1^2+y_1z_1+z_1x_1-x_1y_1)=\tfrac{4}{3}(x_1-y_1+z_1-1)^2$

or $\quad x_1^2+y_1^2+z_1^2+y_1z_1+z_1x_1-x_1y_1=2(x_1^2+y_1^2+z_1^2+1-2x_1y_1$
$\qquad +2x_1z_1-2x_1-2y_1z_1+2y_1-2z_1)$

or $\quad x_1^2+y_1^2+z_1^2-5y_1z_1+3z_1x_1-3x_1y_1-4x_1+4y_1-4z_1+2=0.$

$\therefore$ the locus of $P(x_1, y_1, z_1)$ is given by

$$x^2+y^2+z^2-5yz+3zx-3xy-4x+4y-4z+2=0.$$

Ex. 5. *Find the locus of a point whose distance from x-axis is twice its distance form the yz-plane.* **(Agra 1981)**

Sol. Let the given point be $P(x_1, y_1, z_1)$ whose locus is required to be found.

The equations of x-axis are

$$(x-0)/1=(y-0)/0=(z-0)/0, \quad ...(1)$$

because the x-axis passes through $(0, 0, 0)$ and has d.c.'s 1, 0, 0.

If p_1 be the perpendicular distance of $P(x_1, y_1, z_1)$ from (1), then

$$p_1^2=\begin{vmatrix}0 & 0\\ y_1 & z_1\end{vmatrix}^2+\begin{vmatrix}0 & 1\\ z_1 & x_1\end{vmatrix}^2+\begin{vmatrix}1 & 0\\ x_1 & y_1\end{vmatrix}^2$$

or $\quad p_1^2=(-z_1)^2+(y_1)^2 \quad$ or $\quad p_1^2=y_1^2+z_1^2. \quad ...(2)$

Let p_2 be the perpendicular distance of $P(x_1, y_1, z_1)$ from the yz-plane. Then $\quad p_2=x$-co-ordinate of $P=x_1. \quad ...(3)$

According to the given problem, we have

$$p_1=2p_2 \quad \text{or} \quad p_1^2=4p_2^2$$

or $\quad y_1^2+z_1^2=4x_1^2, \quad$ [using (2) and (3)]

$\therefore$ the locus of the point $P(x_1, y_1, z_1)$ is $4x^2-y^2-z^2=0$.

Ex. 6. *Find the length of the perpendicular drawn from origin to the line $x+2y+3z+4=0=2x+3y+4z+5$. Also find the equations of this perpendicular and the co-ordinates of the foot of the perpendicular.* **(Meerut 1984 S, 84 R)**

Sol. The equations of the given line AB (say) are

$$x+2y+3z+4=0=2x+3y+4z+5.$$

The symmetrical form of the above given line is

$$(x-2)/1=(y+3)/(-2)=(z-0)/1. \quad ...(1)$$

The line AB given by (1) is passing through $A(2, -3, 0)$ and has d.r.'s $1, -2, 1$. Here the point P is $(0, 0, 0)$.

Draw PN perpendicular to AB, so that N, the foot of the perpendicular lies on (1). Hence the co-ordinates of N may be written as $\quad (r+2, -2r-3, r). \quad ...(2)$

The d.r.'s of PN are $\quad r+2-0, -2r-3-0, r-0$

i.e. $\quad r+2, -2r-3, r. \quad ...(3)$

Since PN is perpendicular to AB, we have

$$(r+2).1+(-2r-3)(-2)+r.1=0 \text{ or } r=-4/3.$$

Putting the value of r in (2), we get

$$N\equiv(2/3, -1/3, -4/3).$$

Again putting the value of r in (3), the d.r.'s of PN are

$$2/3, -1/3, -4/3 \quad i.e. \quad 2, -1, -4.$$

$\therefore$ the equations to PN *i.e.* the equations of a line passing through $P(0, 0, 0)$ and having d.r.'s $2, -1, -4$ are

$$\frac{x-0}{2}=\frac{y-0}{-1}=\frac{z-0}{-4} \quad \text{or} \quad \frac{x}{2}=\frac{y}{-1}=\frac{z}{-4}.$$

The length of the perpendicular PN

= the distance between the points P and N

$$=\sqrt{\left\{\left(\frac{2}{3}-0\right)^2+\left(-\frac{1}{3}-0\right)^2+\left(-\frac{4}{3}-0\right)^2\right\}}=\frac{\sqrt{(21)}}{3}.$$

Ex. 7. *Find the equations of the two planes through the origin which are parallel to the line $(x-1)/2=(y+3)/(-1)=(z+1)/(-2)$ and distant 5/3 from it.* **(Meerut 1976, 89, 89 S)**

Sol. Let the equation of any plane through the origin be

$$ax+by+cz=0. \quad ...(1)$$

The equations of the given line are

$$\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z+1}{-2}. \quad ...(2)$$

If the plane (1) is parallel to the line (2), then the normal to (1) whose d.r.'s are a, b, c will be perpendicular to the line (2) and hence we have

$$2.a-1.b-2.c=0 \quad \text{or} \quad 2a-b-2c=0. \quad ...(3)$$

According to the given problem, the plane (1) is at a distance 5/3 from the line (2).

Hence the distance of $(1, -3, -1)$, a point on the line (2), from the plane (1) $\quad =5/3$

or $$\frac{a.1+b.(-3)+c.(-1)}{\sqrt{(a^2+b^2+c^2)}}=\frac{5}{3}.$$

On squaring. $9\,(a-3b-c)^2=25\,(a^2+b^2+c^2)$

or $9\,(a^2+9b^2+c^2-6ab-2ac+6bc)=25\,(a^2+b^2+c^2)$

or $16a^2-56b^2+16c^2+54ab+18ac-56bc=0$

or $8a^2-28b^2+8c^2+27ab+9ac-28bc=0.$ ...(4)

Putting the value of $b=2\,(a-c)$ from (3) in (4), we get

$8a^2-28\times 4\,(a-c)^2+8c^2+27a.2.(a-c)+9ac-2bc.2\,(a-c)=0$

or $-50a^2+125ac-50c^2=0$ or $2a^2-5ac+2c^2=0$

or $(a-2c)\,(2a-c)=0$ or $a=2c,\ \frac{1}{2}c.$

For $a=2c$, from (3). $b=2c$

and for $a=\frac{1}{2}c$, from (3), $b=-c$.

Putting $a=2c$, $b=2c$ in (1), the equation of one plane is

$2cx+2cy+cz=0$ or $2x+2y+z=0.$ ...(5)

Again putting $a=\frac{1}{2}c$, $b=-c$ in (1), the equation of the second plane is $\frac{1}{2}cx-cy+cz=0$ or $x-2y+2z=0.$ ...(6)

The equations (5) and (6) are the equations of the required planes.

§ 11. Intersection of three planes. (M.U. 1990)

Let the equations of three planes be given by

$u_1\equiv a_1x+b_1y+c_1z+d_1=0,$ ...(1)

$u_2\equiv a_2x+b_2y+c_2z+d_2=0,$ ...(2)

and $u_3\equiv a_3x+b_3y+c_3z+d_3=0,$ ...(3)

where u_1, u_2 and u_3 denote respectively the left hand members in the equations (1), (2) and (3). No two of these three planes are parallel.

We know that two non-parallel planes intersect in a straight line and hence we get three lines of intersection by taking two planes at a time out of the three planes given by (1), (2) and (3). There arise following three cases :

Case I. The three lines of intersection explained above may coincide *i.e.* the three given planes have a common line of intersection.

Case II. The three lines of intersection explained above may be parallel to each other and no two of them coincide. In this case the three given planes form a triangular prism.

Case III. The three lines of intersection explained above may intersect in a common point. In this case the three planes intersect in a point.

Before proceeding to prove the actual theorem, for convenience, we make use of some **notations** given as follows :

Consider the matrix (or a rectangular array)

$$\left\|\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix}\right\| \quad ...(i)$$

Let the determinant obtained by omitting the first column in (i) be denoted by Δ_1 *i.e.* we put

$$\Delta_1=\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

Similarly, the determinants obtained by omitting the second, third and fourth columns will be denoted respectively by Δ_2, Δ_3 and Δ_4. Thus we put

$$\Delta_2=\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix},\ \Delta_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

$$\Delta_4=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The symmetrical form of the line of intersection of the planes (1) and (2) is [See § 4]

$$\frac{x-\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)}{b_1c_2-b_2c_1}=\frac{y-\left(\dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}\right)}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1}, \quad ...(4)$$

where $a_1b_2-a_2b_1\neq 0.$

Now we shall discuss the three cases given above in detail as follows :

Case I. The three planes intersect in a common line.

The equation of any plane through the line of intersection of the planes (1) and (2) is given by

$u_1+\lambda u_2=0$

or $(a_1x+b_1y+c_1z+d_1)+\lambda\,(a_2x+b_2y+c_2z+d_2)=0$

or $(a_1+\lambda a_2)\,x+(b_1+\lambda b_2)\,y+(c_1+\lambda c_2)\,z+(d_1+\lambda d_2)=0.$ ...(5)

If the three given planes intersect in a common line, then for some value of λ the plane (5) should represent the plane (3). Thus comparing the coefficients in the equations (5) and (3), we have

$$\frac{a_1+\lambda a_2}{a_3}=\frac{b_1+\lambda b_2}{b_3}=\frac{c_1+\lambda c_2}{c_3}=\frac{d_1+\lambda d_2}{d_3}=\mu \text{ (say)},$$

$$\therefore \quad a_1+\lambda a_2-\mu a_3=0,$$
$$c_1+\lambda b_2-\mu b_3=0,$$
$$c_1+\lambda c_2-\mu c_3=0,$$
and
$$d_1+\lambda d_2-\mu d_3=0.$$

Now we are to eliminate two arbitrary constants λ and μ and this can be done from any three out of the four equations given above. Hence eliminating λ and μ from any three equations taken at a time out of these four equations, we have the conditions as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \text{ i.e. } \Delta_4=0$$

$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}=0 \text{ i.e. } \Delta_3=0$$

$$\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}=0 \text{ i.e } \Delta_2=0$$

$$\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}=0 \text{ i.e. } \Delta_1=0,$$

Hence the three planes will have a common line of intersection of $\Delta_4=0$, $\Delta_3=0$, $\Delta_2=0$ and $\Delta_1=0$.

Alternative method. If the three given planes (1), (2) and (3) intersect in a common line then the line (4) [the line of intersection of the planes (1) and (2)] will lie in the plane (3), so that we have [by § 5 (iii)]

$$a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$$

i.e.,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \text{ i.e., } \Delta_4=0$$

and
$$a_3\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}\right)+c_3.0+d_3=0$$

i.e.,
$$a_3(b_1d_2-b_2d_1)+b_3(d_1a_2-d_2a_1)+d_3(a_1b_2-a_2b_1)=0$$

i.e.,
$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}=0 \text{ i.e., } \Delta_3=0.$$

[**Note.** While writing the symmetrical form of the equations (4) of the line of intersection of the planes (1) and (2), we have taken the point on the line for which $z=0$. If we take the point for which $x=0$ the condition $\Delta_1=0$ is obtained instead of $\Delta_3=0$. Similarly by taking the point for which $y=0$, the condition $\Delta_2=0$ will be obtained.]

Hence the planes (1), (2) and (3) will intersect in a common line if $\Delta_4=0$, $\Delta_3=0$, $\Delta_2=0$ and $\Delta_1=0$.

Case II. The three planes form a triangular prism.

The three planes will form a triangular prism if the line of intersection of any two planes is parallel to the third plane and does not lie in it.

The line of intersection of the planes (1) and (2) is given by (4). The line (4) will be parallel to the plane (3) if

$$a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$$

i.e.,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \text{ i.e., } \Delta_4=0.$$

The line (1) will not lie in the plane (3) if

$$a_3\left(\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}\right)+c_3.0+d_3\neq 0$$

i.e.,
$$a_3(b_1d_2-b_2d_1)+b_3(d_1a_2-d_2a_1)+d_3(a_1b_2-a_2b_1)\neq 0$$

i.e.,
$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}\neq 0 \text{ i.e., } \Delta_3\neq 0.$$

Hence the three planes will form a triangular prism if $\Delta_4=0$ **and** $\Delta_3\neq0$ **or** $\Delta_2\neq0$ **or** $\Delta_1\neq0$.

Case III. The three planes intersect in a point.

The three planes will intersect in a point if the line (4) of intersection of the planes (1) and (2), is neither parallel to nor lie in the plane (3). Rather than the line (4) must meet the plane (3) in a point.

Thus the condition that the three planes meet in a point is that $\Delta_4\neq0$.

Alternative method. Solving the equations (1), (2) and (3) by the method of determinants. [This method is called Cramer's Rule], we have

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}}$$

$$=\frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or $$\frac{x}{\Delta_1}=\frac{-y}{\Delta_2}=\frac{z}{\Delta_3}=\frac{-1}{\Delta_4}$$

or $$x=-\frac{\Delta_1}{\Delta_4},\ y=\frac{\Delta_2}{\Delta_4},\ z=-\frac{\Delta_3}{\Delta_4}. \quad \ldots(6)$$

Hence the three planes will intersect in the point whose co-ordinates are given by (6) if $\Delta_4\neq0$.

Working Rule. Let the three planes be given by the equations (1), (2) and (3). Now proceed as follows :

(1) First evaluate Δ_4. If $\Delta_4\neq0$, then the three planes intersect in a point whose co-ordinates are given by the relations (6) above.

(2) If $\Delta_4=0$, then evaluate Δ_3, Δ_2 and Δ_1.

(i) If $\Delta_3\neq0$ (or $\Delta_1\neq0$ or $\Delta_2\neq0$), then the three planes form a triangular prism.

(ii) If $\Delta_3=0$, $\Delta_2=0$ and $\Delta_1=0$, then the three planes intersect in a common line.

Remark. If $\Delta_4=0$ and $\Delta_3=0$ and at least one of the three common minors $a_1b_2-a_2b_1$, $a_2b_3-a_3b_2$ and $a_1b_3-a_3b_1$ of Δ_4 and Δ_3 is not zero, then it can be proved algebrically that $\Delta_2=0$ and $\Delta_1=0$. Consequently in this case the three planes will have a common line of intersection.

SOLVED EXAMPLES (1)

Ex. 1. *Examine the nature of intersection of the planes*

(i) $5x+2y-4z+2=0,\ 4x-2y-5z-2=0,\ 2x+8y-2z-1=0.$

(ii) $x+2y+3z-6=0,\ 3x+4y+5z-2=0,\ 5x+4y+3z+18=0.$

(iii) $2x+4y+2z-7=0,\ 5x+y-z-9=0,\ x-y-z-6=0.$

Sol. (i) The given planes are

$$5x+2y-4z+2=0 \quad \ldots(1)$$
$$4x-2y-5z-2=0 \quad \ldots(2)$$
$$2x+8y-2z-1=0 \quad \ldots(3)$$

The rectangular array of coefficients is

$$\left\|\begin{matrix} 5 & 2 & -4 & 2 \\ 4 & -2 & -5 & -2 \\ 2 & 8 & -2 & -1 \end{matrix}\right\|$$

We have,

$$\Delta_4=\begin{vmatrix} 5 & 2 & -4 \\ 4 & -2 & -5 \\ 2 & 8 & -2 \end{vmatrix}=\begin{vmatrix} 1 & 2 & -4 \\ -1 & -2 & -5 \\ 0 & 8 & -2 \end{vmatrix},$$

on adding the 3rd column to the 1st column

$$=\begin{vmatrix} 1 & 2 & -4 \\ 0 & 0 & -9 \\ 0 & 8 & -2 \end{vmatrix}, \text{ on adding the first row to the second row}$$

$=72\neq0.$

Hence the given planes intersect in a point.

Adding (1) and (2), we get

$$9x-9z=0 \text{ or } x=z.$$

Putting $x=z$ in (3), we get

$$8y-1=0 \text{ or } y=1/8.$$

Putting $x=z$ in (2), we get

$$-x-2y=2 \text{ or } x=-2y-2. \qquad \ldots(5)$$

Putting the value of y in (5), we get $x=-9/4=z$.

Hence the point of intersection is $(-9/4, 1/8, -9/4)$.

(ii) The rectangular array of coefficients is

$$\left\| \begin{matrix} 1 & 2 & 3 & -6 \\ 3 & 4 & 5 & -2 \\ 5 & 4 & 3 & 18 \end{matrix} \right\|.$$

We have, $\Delta_4=\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \end{vmatrix}$

$$=\begin{vmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -6 & -12 \end{vmatrix}, \text{ by } R_2-3R_1,\ R_3-5R_1$$

$=24-24=0.$

Since $\Delta_4=0$, therefore the three planes either intersect in a line or form a triangular prism.

Now $\Delta_3=\begin{vmatrix} 1 & 2 & -6 \\ 3 & 4 & -2 \\ 5 & 4 & 18 \end{vmatrix}$

$$=\begin{vmatrix} 1 & 2 & -6 \\ 0 & -2 & 16 \\ 0 & -6 & 48 \end{vmatrix}, \text{ by } R_2-3R_1,\ R_3-5R_1$$

$=-96-(-96)=-96+96=0.$

Similarly we find that $\Delta_2=0$ and $\Delta_1=0$.

Hence the three planes intersect in a line.

(iii) The rectangular array of coefficients is

$$\left\| \begin{matrix} 2 & 4 & 2 & -7 \\ 5 & 1 & -1 & -9 \\ 1 & -1 & -1 & -6 \end{matrix} \right\|$$

We have, $\Delta_4=\begin{vmatrix} 2 & 4 & 2 \\ 5 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}$

$$=\begin{vmatrix} 0 & 6 & 4 \\ 1 & 6 & 4 \\ 1 & -1 & -1 \end{vmatrix}, \text{ by } R_2-5R_3,\ R_1-2R_3$$

$=24-24=0.$

Since $\Delta_4=0$, therefore the three planes either intersect in a line or form a triangular prism.

Now $\Delta_3=\begin{vmatrix} 2 & 4 & -7 \\ 5 & 1 & -9 \\ 1 & -1 & -6 \end{vmatrix}=\begin{vmatrix} 2 & 6 & 5 \\ 5 & 6 & 21 \\ 1 & 0 & 0 \end{vmatrix},$

by C_2+C_1, C_3+6C_1

$=1.(126-30)=96\neq 0.$

Hence the three planes form a triangular prism.

Ex. 2. *Show that the planes*

$$2x-3y-7z=0,\ 3x-14y-13z=0,\ 8x-31y-33z=0$$

pass through one line and find its equations. **[Meerut 1977]**

Sol. The rectangular array of coefficients is

$$\left\| \begin{matrix} 2 & -3 & -7 & 0 \\ 3 & -14 & -13 & 0 \\ 8 & -31 & -33 & 0 \end{matrix} \right\|.$$

We have, $\Delta_4=\begin{vmatrix} 2 & -3 & -7 \\ 3 & -14 & -13 \\ 8 & -31 & -33 \end{vmatrix}$

$$=\begin{vmatrix} 2 & -1 & -1 \\ 3 & -11 & -4 \\ 8 & -23 & -9 \end{vmatrix}, \text{ by } C_2+C_1,\ C_3+3C_1$$

$$=\begin{vmatrix} 0 & 0 & -1 \\ -5 & -7 & -4 \\ -10 & -14 & -9 \end{vmatrix}, \text{ by } C_1+2C_3,\ C_2-C_3$$

$$=-1\,(70-70)=0.$$

Since $\Delta_4=0$, therefore, the three planes either intersect in a line or form a triangular prism.

$$\text{Now } \Delta_3=\begin{vmatrix} 2 & -3 & 0 \\ 3 & -14 & 0 \\ 8 & -31 & 0 \end{vmatrix}=0.$$

Similarly $\Delta_2=0$ and $\Delta_1=0$.

Hence the three planes intersect in a common line.

Clearly the three planes pass through (0, 0, 0) and hence the common line of intersection will pass through (0, 0, 0). The equations of the common line are given by any of the two given planes. Therefore the equations of the common line are given by

$$2x-3y-7z=0,$$

and $$3x-14y-13z=0.$$

$\therefore$ the symmetrical form of the line is given by

$$\frac{x}{39-98}=\frac{y}{-21+26}=\frac{z}{-28+9} \text{ or } \frac{x}{-59}=\frac{y}{5}=\frac{z}{-19}.$$

Ex. 3. *Prove that the planes*

$$x-2y+z-3=0,\ x+y-2z-3=0,\ x-z-1=0$$

form a triangular prism.

Sol. Proceeding as in Ex. 1 above, we get $\Delta_4=0$ and $\Delta_3\neq0$. Hence the given planes form a triangular prism.

Ex. 4. *Prove that the planes*

$$x=cy+bz,\ y=az+cx,\ z=bx+ay$$

pass through one line if $a^2+b^2+c^2+2abc=1$, and show that the line of intersection then has the equations

$$\frac{x}{\sqrt{(1-a^2)}}=\frac{y}{\sqrt{(1-b^2)}}=\frac{z}{\sqrt{(1-c^2)}}.$$

[Bundelkhand 1978; Punjab 77; Rajasthan 75, 77]

Sol. The equations of three given planes are

$$x-cy-bz=0,\ cx-y+az=0,\ bx+ay-z=0.$$

The rectangular array of coefficients is

$$\begin{Vmatrix} 1 & -c & -b & 0 \\ c & -1 & a & 0 \\ b & a & -1 & 0 \end{Vmatrix}.$$

$$\text{We have, } \Delta_4=\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix}$$

$$=1\,(1-a^2)+c\,(-c-ab)-b\,(ac+b)$$
$$=-a^2-b^2-c^2-2abc+1.$$

$$\text{Also } \Delta_3=\begin{vmatrix} 1 & -c & 0 \\ c & -1 & 0 \\ b & a & 0 \end{vmatrix}=0.$$

Similarly $\Delta_2=0$ and $\Delta_1=0$.

Hence the given planes intersect in a line if $\Delta_4=0$ *i.e.*, if

$$a^2+b^2+c^2+2abc=1.$$

Clearly the given planes pass through (0, 0, 0) and hence the common line of intersection will pass through (0, 0, 0). Let l, m, n be the d.r.'s of this line. It being perpendicular to the normal of each plane, we have

$$l-cm-bn=0 \quad \ldots(1)$$
$$cl-m+an=0 \quad \ldots(2)$$
$$bl+am-n=0. \quad \ldots(3)$$

Solving (1) and (2), $\dfrac{l}{-ac-b}=\dfrac{m}{-bc-a}=\dfrac{n}{-1+c^2}$

or $$\frac{l}{ac+b}=\frac{m}{bc+a}=\frac{n}{1-c^2}. \quad \ldots(4)$$

Solving (2) and (3), $\dfrac{l}{1-a^2}=\dfrac{m}{ab+c}=\dfrac{n}{ca+b}. \quad \ldots(5)$

Solving (3) and (1), $\dfrac{l}{ab+c}=\dfrac{m}{1-b^2}=\dfrac{n}{bc+a} \quad \ldots(6)$

Taking first two terms of each of (5) and (6) and then multiplying, we get

$$\frac{l^2}{(ab+c)\,(1-a^2)}=\frac{m^2}{(ab+c)\,(1-b^2)} \text{ or } \frac{l^2}{1-a^2}=\frac{m^2}{1-b^2}. \quad \ldots(7)$$

Similarly from (6) and (4), we have

$$\frac{m^2}{1-b^2}=\frac{n^2}{1-c^2}. \quad ...(8)$$

Now from (7) and (8), we get

$$\frac{l^2}{1-a^2}=\frac{m^2}{1-b^2}=\frac{n^2}{1-c^2}.$$

$$\therefore \quad \frac{l}{\sqrt{(1-a^2)}}=\frac{m}{\sqrt{(1-b^2)}}=\frac{n}{\sqrt{(1-c^2)}} \quad ...(9)$$

$\therefore$ the d.r.'s of the common line of intersection of the given planes are given by (9). Since the line passes through the origin, hence, its equations are given by

$$\frac{x}{\sqrt{(1-a^2)}}=\frac{y}{\sqrt{(1-b^2)}}=\frac{z}{\sqrt{(1-c^2)}}.$$

Ex. 5. *Prove that the planes*

$$x+ay+(b+c)\,z+d=0,$$
$$x+by+(c+a)\,z+d=0,$$
$$x+cy+(a+b)\,z+d=0,$$

pass through one line. **[Allahabad 1978, Kanpur 82]**

Sol. The rectangular array (or matrix) is

$$\left\|\begin{matrix} 1 & a & b+c & d \\ 1 & b & c+a & d \\ 1 & c & a+b & d \end{matrix}\right\|.$$

We have,
$$\Delta_4=\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$
$$=\begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix} \quad \text{adding 3rd column to 2nd}$$
$$=(a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}=0.$$

Also
$$\Delta_3=\begin{vmatrix} 1 & a & d \\ 1 & b & d \\ 1 & c & d \end{vmatrix}=d\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}=0.$$

Similarly $\Delta_2=0$ and $\Delta_1=0$.

Since $\Delta_4=0$, $\Delta_3=0$, $\Delta_2=0$ and $\Delta_1=0$, therefore the given planes intersect in a line.

Ex. 6. *Show that the planes*

$$ny-mz=\lambda,\ lz-nx=\mu \text{ and } mx-ly=\nu$$

have a common line if $l\lambda+m\mu+n\nu=0$, and the direction ratios of the line are l, m, n. **[Punjab 1980; Behrampur 76 S]**

Show further that the distance of the line from the origin is

$$\{(\lambda^2+\mu^2+\nu^2)/(l^2+m^2+n^2)\}^{1/2}.$$

Sol. The equations of the given planes may be written as

$$0.x+ny-mz-\lambda=0 \quad ...(1)$$
$$-nx+0.y+lz-\mu=0 \quad ...(2)$$
$$mx-ly+0.z-\nu=0. \quad ...(3)$$

The rectangular array (or matrix) of coefficients is

$$\left\|\begin{matrix} 0 & n & -m & -\lambda \\ -n & 0 & l & -\mu \\ m & -l & 0 & -\nu \end{matrix}\right\|.$$

We have,
$$\Delta_4=\begin{vmatrix} 0 & n & -m \\ -n & 0 & l \\ m & -l & 0 \end{vmatrix}$$
$$=0-n\,(0-lm)-m\,(nl-0)=0.$$

$\therefore$ the given planes will either meet in a line or form a triangular prism.

Now
$$\Delta_3=\begin{vmatrix} 0 & n & \lambda \\ -n & 0 & -\mu \\ m & -l & -\nu \end{vmatrix}$$
$$=0-n\,(n\nu+m\mu)-\lambda\,(nl-0)$$
$$=-n\,(l\lambda+m\mu+n\nu).$$

If the planes are to meet in a line then Δ_3 must also be zero. Then we have $\quad n\,(l\lambda+m\mu+n\nu)=0$

or

$$l\lambda+m\mu+n\nu=0. \quad ...(4)$$

[We have assumed $n\neq 0$.]

If the condition (4) is satisfied, then we can see that $\Delta_2=0$

and also $\Delta_1=0$. Hence (4) is the condition for the three given planes to have a common line of intersection.

Let a, b, c be the d.r.'s of the line of intersection. It being perpendiculer to the normal of each plane, we have

$$0\,a+nb-mc=0 \qquad \ldots(5)$$
$$-na+0.b+lc=0 \qquad \ldots(6)$$
$$ma-lb+0.c=0. \qquad \ldots(7)$$

Solving (5) and (6),

$$\frac{a}{nl}=\frac{b}{mn}=\frac{c}{n^2} \text{ or } \frac{a}{l}=\frac{b}{m}=\frac{c}{n},$$

showing that the d.r.'s of the line of intersection are l, m, n.

To find the equations of the line of intersection. The line of intersection meets the plane $z=0$ in the point given by

$$ny-\lambda=0, \quad -nx-\mu=0, \qquad \text{[Putting } z=0 \text{ in (1) and (2)]}$$

or $\qquad x=-\mu/n,\ y=\lambda/n.$

$\therefore$ the line of intersection passes through the point $(-\mu/n, \lambda/n, 0)$ and has d.r.'s l, m, n and therefore its equations are given by

$$\frac{x+\mu/n}{l}=\frac{y-\lambda/n}{m}=\frac{z-0}{n}. \qquad \ldots(8)$$

The distance of the line (8) from the origin. Using § 10, the required distance p is given by

$$p^2=\frac{1}{(l^2+m^2+n^2)}\left\{\begin{vmatrix} m & n \\ 0-\lambda/n & 0-0\end{vmatrix}^2 + \begin{vmatrix} n & l \\ 0-0 & 0+\mu/n\end{vmatrix}^2 + \begin{vmatrix} l & m \\ 0+\mu/n & 0-\lambda/n\end{vmatrix}^2\right\}$$
$$=\frac{1}{(l^2+m^2+n^2)}\left\{\lambda^2+\mu^2+\left(\frac{-l\lambda-m\mu}{n}\right)^2\right\}$$
$$=\frac{1}{(l^2+m^2+n^2)}\left\{\lambda^2+\mu^2+\left(\frac{n\nu}{n}\right)^2\right\} \qquad \text{[using (4)]}$$

$\therefore \quad p=\{(\lambda^2+\mu^2+\nu^2)/(l^2+m^2+n^2)\}^{1/2}.$

Ex. 7. *Prove that the planes $x=y \sin \psi+z \sin \phi$, $y=z \sin \theta+x \sin \psi$, and $z=x \sin \phi+y \sin \theta$ will intersect in the line*

$$\frac{x}{\cos \theta}=\frac{y}{\cos \phi}=\frac{z}{\cos \psi} \quad \textit{if } \theta+\phi+\psi=\tfrac{1}{2}\pi.$$

[Allahabad 1982; Garhwal 78 S]

Sol. The equations of the planes may be written as

$$x-y \sin \psi-z \sin \phi=0, \qquad \ldots(1)$$
$$x \sin \psi-y+z \sin \theta=0, \qquad \ldots(2)$$
$$x \sin \phi+y \sin \theta-z=0. \qquad \ldots(3)$$

Let us find the line of intersection of the planes (1) and (2). Let l, m, n be the direction cosines of the line of intersection of the planes (1) and (2). It being perpendicular to the normals of both the planes, we have

$$l-m \sin \psi-n \sin \phi=0, \quad l \sin \psi-m+n \sin \theta=0.$$

Solving, we get

$$\frac{l}{-\sin \psi \sin \theta-\sin \phi}=\frac{m}{-\sin \phi \sin \psi-\sin \theta}=\frac{n}{-1+\sin^2 \psi}$$

or $$\frac{l}{\sin \psi \sin \theta+\sin \phi}=\frac{m}{\sin \phi \sin \psi+\sin \theta}=\frac{n}{1-\sin^2 \psi}. \qquad \ldots(4)$$

If $\theta+\phi+\psi=\frac{1}{2}\pi$, then we have

$$\sin \theta=\sin \{\tfrac{1}{2}\pi-(\phi+\psi)\}=\cos (\phi+\psi)$$
$$=\cos \phi \cos \psi-\sin \phi \sin \psi$$

or $$\sin \theta+\sin \phi \sin \psi=\cos \phi \cos \psi. \qquad \ldots(5)$$

Similarly $\sin \phi+\sin \psi \sin \theta=\cos \psi \cos \theta. \qquad \ldots(6)$

Using the relations (5) and (6), (4) becomes,

$$\frac{l}{\cos \psi \cos \theta}=\frac{m}{\cos \psi \cos \phi}=\frac{n}{\cos^2 \psi}$$

or $$\frac{l}{\cos \theta}=\frac{m}{\cos \phi}=\frac{n}{\cos \psi}. \qquad \ldots(7)$$

Clearly the planes (1) and (2) pass through (0, 0, 0) and so their line of intersection will pass through (0, 0, 0) so that its equations are given by

$$x/\cos \theta=y/\cos \phi=z/\cos \psi. \qquad \ldots(8)$$

Now it remains to prove that the line (8) must lie on the plane (3).

The point (0, 0, 0) through which the line (8) passes also lies on the plane (3). Also the normal to the plane (3) whoss d.r.'s are $\sin \phi$, $\sin \theta$, -1 must be perpendicular to (8), the condition for which is

$$\cos \theta \sin \phi+\cos \phi \sin \theta+\cos \psi\,(-1)=0$$

or $\quad \sin (\theta+\phi)-\cos \psi=0$, or $\sin (\frac{1}{2}\pi-\psi)-\cos \psi=0$

or $\quad \cos \psi-\cos \psi=0$ or $0=0$ which is true.

Hence the line (8) also lies on the plane (3). Thus the equations (8) are the equations of the required line.

Ex. 8. *Show that the planes $ax+hy+gz=0$, $hx+by+fz=0$, $gx+fy+cz=0$ have a common line of intersection if*

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

and the direction ratios of the line satisfy the equations

$$\frac{l^2}{\frac{\partial\Delta}{\partial a}}=\frac{m^2}{\frac{\partial\Delta}{\partial b}}=\frac{n^2}{\frac{\partial\Delta}{\partial c}}.$$

Sol. The equations of the given planes are

$ax+hy+gz=0$ ...(1), $hx+by+fz=0$ ...(2)

$gx+fy+cz=0.$ ...(3)

The rectangular array of coefficients is

$$\begin{Vmatrix} a & h & g & 0 \\ h & b & f & 0 \\ g & f & c & 0 \end{Vmatrix}.$$

We have

$$\Delta_4=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=\Delta \text{ (as given in the problem)}$$

or $\Delta_4=\Delta=abc+2fgh-af^2-bg^2-ch^2.$ ...(4)

Also

$$\Delta_3=\begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & 0 \end{vmatrix}=0. \text{ Similarly } \Delta_2=0 \text{ and } \Delta_1=0.$$

If the given planes intersect in a line then Δ_4 must be zero (as Δ_3, Δ_2 and Δ_1 are already zero). Hence the given planes will have a common line of intersection iff $\Delta_4=0$ or $\Delta=0$

or $abc+2fgh-af^2-bg^2-ch^2=0.$ ...(5)

Let l, m, n be the d.r.'s of the common line of intersection of the given planes. It being perpendicular to the normals of the planes, we have

$$al+hm+gn=0,\ hl+bm+fn=0,\ gl+fm+cn=0.$$

Solving any two (say the first two) of these relations, we get

$$\frac{l}{hf-bg}=\frac{m}{gh-af}=\frac{n}{ab-h^2}.$$

Squaring, $\frac{l^2}{(hf-bg)^2}=\frac{m^2}{(gh-af)^2}=\frac{n^2}{(ab-h^2)^2}.$ ...(6)

Differentiating (4) partially w.r.t. a, b, c respectively, we have

$\partial\Delta/\partial a=bc-f^2$, $\partial\Delta/\partial b=ac-g^2$, $\partial\Delta/\partial c=ab-h^2$. ...(7)

Now $(hf-bg)^2=h^2f^2+b^2g^2-2bfgh$

$=h^2f^2+b^2g^2-b(af^2+bg^2+ch^2-abc)$ using (5)

$=h^2f^2+b^2g^2-abf^2-b^2g^2-bch^2+ab^2c$

$=h^2f^2-abf^2-bch^2+ab^2c$

$=-f^2(ab-h^2)+bc(ab-h^2)$

$=(ab-h^2)(bc-f^2).$

Similarly $(gh-af)^2=(ab-h^2)(ca-g^2)$.

Substituting these values in (6), we get

$$\frac{l^2}{(ab-h^2)(bc-f^2)}=\frac{m^2}{(ab-h^2)(ca-g^2)}=\frac{n^2}{(ab-h^2)^2}$$

or $$\frac{l^2}{bc-f^2}=\frac{m^2}{ca-g^2}=\frac{n^2}{ab-h^2}$$

or $$\frac{l^2}{\frac{\partial\Delta}{\partial a}}=\frac{m^2}{\frac{\partial\Delta}{\partial b}}=\frac{n^2}{\frac{\partial\Delta}{\partial c}}$$ [using (7)]

Ex. 9. *For what values of k do the planes*

$$x-y+z+1=0,\ kx+3y+2z-3=0,\ 3x+ky+z-2=0$$

(i) intersect in a point ; (ii) intersect in a line ; (iii) form a triangular prism ?

Sol. The rectangular array of coefficients is

$$\begin{Vmatrix} 1 & -1 & 1 & 1 \\ k & 3 & 2 & 3 \\ 3 & k & 1 & -2 \end{Vmatrix}$$

Now we calculate the following determinants :

$$\Delta_4=\begin{vmatrix} 1 & -1 & 1 \\ k & 3 & 2 \\ 3 & k & 1 \end{vmatrix}=\begin{vmatrix} 0 & -1 & 0 \\ k+3 & 3 & 5 \\ 3+k & k & k+1 \end{vmatrix}$$

adding 2nd column to 1st and 3rd

$$=(k+3)\begin{vmatrix} 0 & -1 & 0 \\ 1 & 3 & 5 \\ 1 & k & k+1 \end{vmatrix}=(k+3)(k+1-5)$$

$=(k+3)(k-4).$

$$\Delta_4=\begin{vmatrix}1 & -1 & 1\\ k & 3 & -3\\ 3 & k & -2\end{vmatrix}=\begin{vmatrix}0 & -1 & 0\\ k+3 & 3 & 0\\ 3+k & k & k-2\end{vmatrix},$$ adding 2nd column to 1st and 3rd

$$=(k+3)(k-2),$$

$$\Delta_3=\begin{vmatrix}1 & 1 & 1\\ k & 2 & -3\\ 3 & 1 & -2\end{vmatrix}=\begin{vmatrix}0 & 1 & 0\\ k-2 & 2 & -5\\ 2 & 1 & -3\end{vmatrix},$$

adding (−1) times 2nd column to 1st and 3rd

$$=-\{(k-2)(-3)+10\}=3k-16,$$

and $$\Delta_1=\begin{vmatrix}-1 & 1 & 1\\ 3 & 2 & -3\\ k & 1 & -2\end{vmatrix}=\begin{vmatrix}0 & 1 & 1\\ 0 & 2 & -3\\ k-2 & 1 & -2\end{vmatrix},$$

adding 3rd column to 1st

$$=-5(k-2).$$

(i) The given planes will intersect in a point if $\Delta_4\neq 0$ and so we must have $k\neq -3$ and $k\neq 4$. Thus the given planes will intersect in a point for all real values of k other than -3 and 4.

(ii) If $k=-3$, we have $\Delta_4=0$, $\Delta_3=0$ but $\Delta_2\neq 0$. Hence the given planes will form a triangular prism if $k=-3$.

(iii) If $k=4$, we have $\Delta_4=0$ but $\Delta_3\neq 0$. Hence the given planes will form a triangular prism if $k=4$.

We observe that for no value of k the given planes will have a common line of intersection.

Ex. **10**. *The plane $x/a+y/b+z/c=1$ meets the axes in A, B and C. Prove that the planes through the axes and the internal bisectors of the angles of the triangle ABC pass through the line*

$$\frac{x}{a\sqrt{(b^2+c^2)}}=\frac{y}{b\sqrt{(c^2+a^2)}}=\frac{z}{c\sqrt{(a^2+b^2)}}.$$

Sol. Plane $x/a+y/b+z/c=1$ meets the axes in $A\ (a, 0, 0)$, $B\ (0, b, 0)$, $C\ (0, 0, c)$.

The equation of any plane through x-axis is (*i.e.* $y=0$, $z=0$) is

$$y+\lambda z=0. \quad \ldots(1)$$

The d.r.'s of AB are $-a, b, 0$

$\therefore$ D.C.'s of AB are $\dfrac{-a}{\sqrt{(a^2+b^2)}}, \dfrac{b}{\sqrt{(a^2+b^2)}}, 0$

D.C.'s of AC $\dfrac{-a}{\sqrt{(a^2+c^2)}}, 0, \dfrac{c}{\sqrt{(a^2+c^2)}}$.

D.C.'s of interior bisector of $\angle BAC$ are

$$-\frac{a}{2}\left(\frac{1}{\sqrt{(a^2+b^2)}}+\frac{1}{\sqrt{(a^2+c^2)}}\right), \frac{b}{2\sqrt{(a^2+b^2)}}, \frac{c}{2\sqrt{(a^2+c^2)}}$$

Now if plane (1) passes through the internal bisector of $\angle BAC$, then normal of (1) will be perpendicular to the internal bisector of $\angle BAC$.

$$\therefore\quad 0+1.\frac{b}{2\sqrt{(a^2+b^2)}}+\lambda.\frac{c}{2\sqrt{(a^2+c^2)}}=0$$

or $$\lambda=-\frac{b\sqrt{(a^2+c^2)}}{c\sqrt{(a^2+b^2)}}$$

Put this value of λ in (1), the equation of plane through x-axis and internal bisector of $\angle BAC$ is

$$y-\frac{b\sqrt{(a^2+c^2)}}{c\sqrt{(a^2+b^2)}}z=0$$

or $$\frac{y}{b\sqrt{(a^2+c^2)}}=\frac{z}{c\sqrt{(a^2+b^2)}} \quad \ldots(2)$$

Similarly the equations of other two planes are

$$\frac{x}{a\sqrt{(b^2+c^2)}}=\frac{y}{b\sqrt{(a^2+c^2)}} \quad \ldots(3)$$

$$\frac{z}{c\sqrt{(a^2+b^2)}}=\frac{x}{a\sqrt{(b^2+c^2)}}. \quad \ldots(4)$$

$\therefore$ Planes (2), (3), (4) pass through the lines

$$\frac{x}{a\sqrt{(b^2+c^2)}}=\frac{y}{b\sqrt{(a^2+c^2)}}=\frac{z}{c\sqrt{(a^2+b^2)}}.$$

Exercises

1. Find the coordinates of the point where the line
$$(x-1)/2=(y-2)/-3=(z+3)/4$$
meets the plane $2x+4y-z+1=0$.
Ans. $(10/3, -3/2, 5/3)$.

2. Show that the distance of the point of intersection of the line
$$(x-3)/1=(y-4)/2=(z-5)/2$$
and the plane $x+y+z=17$ from the point $(3, 4, 5)$ is 3.

3. Find in symmetrical form the equations of the line
$$x+y+z+1=0=4x+y-2z+2$$

and find its direction cosines.

Ans. $$\frac{x+\frac{1}{3}}{-1}=\frac{y+\frac{2}{3}}{2}=\frac{z-0}{-1}.$$

Direction cosines are $-1/\sqrt{6},\ 2/\sqrt{6},\ -1/\sqrt{6}$.

4. Show that the equation of the plane which contains the two parallel lines

$$\frac{x-4}{1}=\frac{y-3}{-4}=\frac{z-2}{5} \text{ and } \frac{x-3}{1}=\frac{y+2}{-4}=\frac{z}{5}$$

is $11x-y-3z-35=0$.

5. Show that the equation of the plane containing the line

$$\frac{y}{b}+\frac{z}{c}=1,\ x=0$$

and parallel to the line $x/a-z/c=1,\ y=0$ is

$$\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0.$$

6. Find the equations of the perpendicular from the point (1, 6, 3) to the line

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}.$$

Find also the coordinates of the foot of the perpendicular.

Ans. Coordinates of the foot of the perpendicular are (1, 3, 5).

Equations of the perpendicular are

$$\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}.$$

7. Show that the following pairs of lines are coplanar :

(i) $x-4=-\frac{1}{2}(y+1)=z$

and $4x-y+5z-7=0=2x-5y-z-3$.

Also find the equation of the plane containing them.

(Kanpur 1980)

(iii) $\frac{1}{3}(x-3)=-\frac{1}{4}(y-2)=(z+1)$

and $x+2y+3z=0=2x+4y+3z+3$.

Also find the point of intersection.

Ans. (i) $x+2y+3z=2$ (ii) $(9, -6, 1)$.

8. Show that the lines

$x+2y-5z+9=0=3x-y+2z-5$

and $2x+3y-z-3=0=4x-5y+z+3$ are coplanar.

9. **Prove that the lines**

$x-3y+2z+4=0=2x+y+4z+1$

and $3x+2y+5z-1=0=2y+z$

intersect at the point $(3, 1, -2)$.

10. Find the equations of the line which can be drawn from the point $(2, -1, 3)$ to intersect the lines

$\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$

and $\frac{1}{4}(x-4)=\frac{1}{5}y=\frac{1}{3}(z+3)$.

Ans. $12x+4y-9z+7=0=11x-10y+2z-38$.

11. Find the distance of $(-2, 1, 5)$ from the line through $(2, 3, 5)$ whose direction cosines are proportional to $2, -3, 6$.

Ans. $\frac{4}{7}\sqrt{(61)}$.

12. Prove that the equations of the perpendicular from the point (1, 6, 3) to the line

$$x=\frac{y-1}{2}=\frac{z-2}{3} \text{ are } \frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}$$

and the coordinates of the foot of the perpendicular are (1, 3, 5). **(Meerut 1977)**

13. Find the locus of a point which moves so that its distance from the line $x=y=z$ is twice its distance from the plane $x+y+z=1$.

Ans. $x^2+y^2+z^2+5xy+5yz+5zx-4x-4y-4z+2=0$.

5

Shortest Distance

Definitions.

§ 1. Skew lines.

Two lines are called *skew lines* or *non-intersecting lines* if they do not lie in the same plane. Skew lines never intersect and are not parallel.

Shortest distance. The straight line which is perpendicular to each of the two skew lines is called the line of shortest distance. The length of the line of shortest distance intercepted between the skew lines is called the length of the shortest distance. The shortest distance is briefly written as S. D.

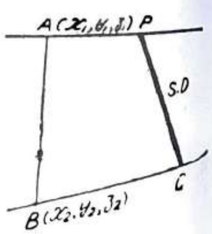
§ 2. Length the equations of the line of shortest distance.

To find the shortest distance between two given lines and to obtain the equations of the shortest distance.

(Allahabad 1978 ; Behrampur 81; Gauhati 76; Indore 78; Kanpur 77. 78, 83; Punjab 76; Rajasthan 73, 77; Rohilkhand 81)

Several methods, depending upon the forms of the equations of the skew lines, are followed to find the shortest distance. They are as follows :

Method I. Projection method. *The equations of the skew lines being given in symmetrical forms.*



Let the equations of the given lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad ...(1)$$

and $$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}. \quad ...(2)$$

The line (1) is passing through the point $A(x_1, y_1, z_1)$ and has d.c.'s proportional to l_1, m_1, n_1. The line (2) is passing through the point $B(x_2, y_2, z_2)$ and has d.c.'s proportional to l_2, m_2, n_2.

Let PQ be the line of shortest distance between the two lines so that PQ is perpendicular to both the lines (1) and (2).

Let l, m, n be the d.c.'s of the line of shortest distance PQ. Then we have

$$ll_1+mm_1+nn_1=0 \quad \text{and} \quad ll_2+mm_2+nn_2=0.$$

Solving, we get

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$
$$=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}}}=\frac{1}{\sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}}}.$$

Now PQ = the length of shortest distance between the given lines (1) and (2)

= the projection of the segment AB on the line of shortest distance PQ

$$=l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$$

$$=\frac{(m_1n_2-m_2n_1)(x_2-x_1)+(n_1l_2-n_2l_1)(y_2-y_1)+(l_1m_2-l_2m_1)(z_2-z_1)}{\sqrt{\{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2\}}}$$

$$=\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}} \quad ...(3)$$

This is the required length of the S.D. between the given lines (1) and (2).

The equations of the shortest distance. Clearly the line PQ of the shortest distance is coplanar with both the given lines (1) and (2). Hence the line PQ of shortest distance is the line of intersection of the two planes, namely, (i) the plane containing the given line (1) and the line PQ of shortest distance and (ii) the plane containing the given line (2) and the line PQ of the shortest distance.

Now the equation of the plane containing the given line (1) and the line PQ of the shortest distance whose d.c.'s are l, m, n is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix}=0. \quad ...(4)$$

Also the equation of the plane containing the given line (3) and the line PQ of the shortest distance whose d.c.'s are l, m, n is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}=0. \quad ...(5)$$

The equations of the line PQ of the shortest distance are given by the planes (4) and (5) taken together.

Note. If the shortest distance $PQ=0$, then the two given lines (1) and (2) will intersect *i.e.*, they will be coplanar. From (3), we observe that

$$PQ=0 \text{ if } \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

which is also the condition for the two lines (1) and (2) to be coplanar. [See chapter 4 § 8 (A)].

Hence we can give an another statement for the two lines to be coplanar.

"*Two lines are coplanar if the shortest distance between them vanishes.*"

Method II. General co-ordinates. *The equations of the two lines being given in symmetrical form* :

Let the equations of the two lines be given by (1) and (2) [See method I].

The general co-ordinates of the points on the two lines (1) and (2) are given by

$(l_1r_1+x_1,\ m_1r_1+y_1,\ n_1r_1+z_1)$, say the point P.

and $(l_2r_2+x_2,\ m_2r_2+y_2,\ n_2r_2+z_2)$, say the point Q.

Let P and Q be the points where the line of shortest distance meets the given lines (1) and (2) respectively, so that the line PQ is perpendicular to both the given lines (1) and (2).

Now find the d.r.'s of the line PQ and apply the conditions that the line PQ is perpendicular to both the given lines (1) and (2). Thus two equations in r_1 and r_2 are obtained. Solve these equations to get r_1 and r_2. Having found r_1 and r_2, the co-ordinates of the points P and Q and also the d.r.'s of the line PQ are known. Now we can at once find the length PQ of the shortest distance and also the equations of PQ.

Method III. *The equations of one line being given in general form and those of the other line in symmetrical form.*

Let the equations of one line be $u_1=0=v_1$ .. (6)

and the equations of second line be

$$(x-x_1)/l_1=(y-y_1)/m_1=(z-z_1)/n_1. \quad ...(7)$$

The equation of any plane through the line (6) is

$$u_1+\lambda v_1=0. \quad ...(8)$$

Find λ so that the plane (8) is parallel to the line (7) and substitute this value of λ in (8). Then the length of shortest distance between the given lines (6) and (7) is equal to the length of the perpendicular from any point, say the point (x_1, y_1, z_1), on the line (7) to the plane (8).

The equations of the shortest distance. The shortest distance is the line of intersection of the two planes namely (a) the plane containing the given line (6) and perpendicular to the plane (8) and (b) the plane containing the given line (7) and perpendicular to the plane (8). Hence the equations of the line of shortest distance are given by the equations of the planes (a) and (b) taken together.

Method IV. *The equations of both lines being given in general form.*

Let the equations of the two given lines be

$$u_1=0=v_1 \quad ...(9) \quad \text{and} \quad u_2=0=v_2. \quad ...(10)$$

The equation of any plane through the line (9) is

$$u_1+\lambda_1v_1=0. \quad ...(11)$$

The equation of any plane through the line (10) is

$$u_2+\lambda_2v_2=0. \quad ...(12)$$

Now λ_1 and λ_2 are determined with the conditions that *the planes* (11) *and* (12) *are parallel.*

$\therefore$ The shortest distance is equal to the distance between the parallel planes (11) and (12).

The equations of the shortest distance are given by the equations of the two planes namely (a) the plane through the line (9) and perpendicular to the plane (11) [or (12)] and (b) the plane through the line (10) and perpendicular to the plane (11) [or (12)].

Remark. For the sake of convenience we can reduce the equations of one or both the straight lines to symmetrical form and then follow the methods I, II or III as explained above.

SOLVED EXAMPLES

Ex. 1. *Find the shortest distance between the lines*

$$(x-1)/2=(y-2)/3=(z-3)/4;$$
$$(x-2)/3=(y-4)/4=(z-5)/5.$$

Show also that the equations of the shortest distance are

$$11x+2y-7z+6=0=7x+y-5z+7.$$

(Agra 1974, 78; Berahmpur 76S, 81S; Madras 76; Meerut 78, 86, 89; Vikram 78)

Sol. The equations of the given lines are

$$(x-1)/2=(y-2)/3=(z-3)/4=r_1 \text{ (say)} \quad ...(1)$$
$$(x-2)/3=(y-4)/4=(z-5)/5=r_2 \text{ (say)}. \quad ...(2)$$

Method I. (Projection method). Let l, m, n be the d.c.'s of the line of S. D. Since it is perpendicular to both the given lines (1) and (2), therefore we have

$$2l+3m+4n=0;\ 3l+4m+5n=0.$$

Solving, we get $\dfrac{l}{15-16}=\dfrac{m}{12-10}=\dfrac{n}{8-9}$

or $\dfrac{l}{-1}=\dfrac{m}{2}=\dfrac{n}{-1}=\dfrac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(-1)^2+(2)^2+(-1)^2\}}}=\dfrac{1}{\sqrt{6}}.$

∴ The d.c.'s of S. D. are $-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}$.

Now A (1, 2, 3) is a point on line (1) and B (2, 4, 5) is a point on the line (2). Hence

the length of S.D. = the projection of join of A and B on the line whose d.c.'s are $-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}$

$$=(-1/\sqrt{6})(2-1)+(2/\sqrt{6})(4-2)+(-1/\sqrt{6})(5-3)$$
$$=1/\sqrt{6}.$$

The equations of S. D. (See § 2)

The equations of the plane through the line (1) and S. D. is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix}=0 \text{ or } 11x+2y-7z+6=0 \quad ...(3)$$

And the equation of the plane through the line (2) and S.D. is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}=0 \text{ or } 7x+y-5z+7=0 \quad ...(4)$$

The equations (3) and (4) together are the equations of the S.D.

Method 2. Any point on line (1) is

$$(2r_1+1, 3r_1+2, 4r_1+3), \text{ say } P. \quad ...(3)$$

Any point on line (2) is

$$(3r_2+2, 4r_2+4, 5r_2+5), \text{ say } Q. \quad ...(4)$$

The d.r.'s of the line PQ are

$$(3r_2+2)-(2r_1+1), (4r_2+4)-(3r_1+2), (5r_2+5)-(4r_1+3)$$

or $$3r_2-2r_1+1, 4r_2-3r_1+2, 5r_2-4r_1+2. \quad ...(5)$$

Let the line PQ be the line of shortest distance, so that PQ is perpendicular to both the given lines (1) and (2) and, therefore, we have

$$2(3r_2-2r_1+1)+3(4r_2-3r_1+2)+4(5r_2-4r_1+2)=0$$

and $$3(3r_2-2r_1+1)+4(4r_2-3r_1+2)+5(5r_2-4r_1+2)=0$$

or $$38r_2-29r_1+16=0 \text{ and } 50r_2-38r_1+21=0.$$

Solving these equations, we get $r_1=1/3, r_2=-1/6$.

Substituting the values of r_1 and r_2 in (3), (4) and (5), we have

$$P(5/3, 3, 13/3), Q(3/2, 10/3, 25/6)$$

and d.r.'s of PQ (the line of S.D.) are $-\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}$ *i.e.* $-1, 2, -1$.

The length of S.D. = the distance between the points P and Q

$$=\sqrt{\left\{\left(\frac{3}{2}-\frac{5}{3}\right)^2+\left(\frac{10}{3}-3\right)^2+\left(\frac{25}{6}-\frac{13}{3}\right)^2\right\}}$$
$$=\sqrt{\left\{\left(-\frac{1}{6}\right)^2+\left(\frac{1}{3}\right)^2+\left(-\frac{1}{6}\right)^2\right\}}=\frac{1}{\sqrt{6}}.$$

The equations of S.D. are either given by equations (3) and (4) of method 1 above or we can write the equations of a line passing through the point P and having d.r.'s $-1, 2, -1$.

Ex. 2 *Find the shortest distance between the lines*

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}, \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}.$$

Find also its equations and the points in which it meets the given lines.

(Avadh 1982; Garhwal 79; Indore 76, 79; Kanpur 74, 80; Lucknow 80, 81; Madrad 78; Meerut 72, 80, 83, 89S; Rohilkhand 78)

Sol. The equations of the given lines are

$$(x-3)/3=(y-8)/-1=(z-3)/1=r_1 \text{ (say)} \quad ...(1)$$

and $$(x+3)/(-3)=(y+7)/2=(z-6)/4=r_2 \text{ (say)} \quad ...(2)$$

Any point on line (1) is $(3r_1+3, -r_1+8, r_1+3)$, say P. ...(3)

Any point on line (2) is $(-3r_2-3, 2r_2-7, 4r_2+6)$, say Q. ...(4)

The d.r.'s of the line PQ are

$$(-3r_2-3)-(3r_1+3), (2r_2-7)-(-r_1+8), (4r_2+6)-(r_1+3)$$

or $$-3r_2-3r_1-6, 2r_2+r_1-15, 4r_2-r_1+3. \quad ...(5)$$

Let the line PQ be the lines of S.D., so that PQ is perpendicular to both the given lines (1) and (2), and so we have

$$3(-3r_2-3r_1-6)-1(2r_2+r_1-15)+1.(4r_2-r_1+3)=0$$

and $-3(-3r_2-3r_1-6)+2.(2r_2+r_1-15)+4(4r_2-r_1+3)=0$

or $\quad -7r_2-11r_1=0$ and $11r_2+7r_1=0$.

Solving these equations, we get $r_1=r_2=0$.

Substituting the values of r_1 and r_2 in (3), (4) and (5), we have

$P(3, 8, 3)$, $Q(-3, -7, 6)$ **Ans.**

And the d.r.'s of PQ (the line of S.D.) are $-6, -15, 3$ or $-2, -5, 1$.

The length of S.D. = the distance between the points P and Q

$$=\sqrt{\{(-3-3)^2+(-7-8)^2+(6-3)^2\}}=3\sqrt{30}.$$

Now the line PQ of shortest distance is the line passing through $P(3, 8, 3)$ and having d.r.'s $-2, -5, 1$ and hence its equations are given by

$$\frac{x-3}{-2}=\frac{y-8}{-5}=\frac{z-3}{1} \text{ or } \frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{1}.$$

Ex. 3. *Find the length of the shortest distance between the lines*

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1};\ \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{-1}.$$

(Garhwal 1981; Kanpur 83; Meerut 83S, 86P; Rohilkhand 82)

Show also that its equations are given by

$(x-1)/2=(y-2)/3=(z-3)/4$. **(Meerut 1986P)**

Sol. The equations of the given lines are

$(x-3)/1=(y-5)/(-2)=(z-7)/1=r_1$ (say); ...(1)

and $(x+1)/7=(y+1)/(-6)=(z+1)/1=r_2$ (say). ...(2)

Any point on line (1) is $(r_1+3, -2r_1+5, r_1+7)$, say P. ...(3)

Any point on line (2) is $(7r_2-1, -6r_2-1, r_2-1)$, say Q. ...(4)

The d.r.'s of the line PQ are

$(7r_2-1)-(r_1+3), (-6r_2-1)-(-2r_1+5), (r_2-1)-(r_1+7)$

or $\quad 7r_2-r_1-4, -6r_2+2r_1-6, r_2-r_1-8$. ...(5)

Let the line PQ be the line of S.D., so that PQ is perpendicular to both the given lines (1) and (2), and so we have

$$1(7r_2-r_1-4)-2(-6r_2+2r_1-6)+1(r_2-r_1-8)=0$$

and $7(7r_2-r_1-4)-6(-6r_2+2r_1-6)+1.(r_2-r_1-8)=0$

or $\quad 20r_2-6r_1=0$ and $86r_2-20r_1=0$.

Solving these equations, we get $r_1=r_2=0$.

Substituting the values of r_1 and r_2 in (3), (4) and (5), we have

$P(3, 5, 7)$, $Q(-1, -1, -1)$

and the d.r.'s of PQ (the line of S.D.) are $-4, -6, -8$ or $2, 3, 4$,

The length of S.D. $=PQ=\sqrt{\{(-1-3)^2+(-1-5)^2+(-1-7)^2\}}$
$=\sqrt{\{(4)^2+(6)^2+(8)^2\}}=2\sqrt{(29)}$.

Now the line PQ of shortest distance is a line passing through $P(3, 5, 7)$ and having d.r.'s 2, 3, 4 and hence its equations are given by

$$\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$$

or $$\frac{x-3}{2}+1=\frac{y-5}{3}+1=\frac{z-7}{4}+1$$

or $$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}.$$ **Proved.**

Ex. 4. *Find the shortest distance between the lines*
$(x-1)/2=(y-2)/3=(z-3)/4$; $(x-2)/3=(y-3)/4=(z-4)/5$.
Hence show that the lines are coplanar. **(Meerut 1985)**

Sol. We are solving this problem by projection method. The equations of the given lines are

$(x-1)/2=(y-2)/3=(z-3)/4$; ...(1)

and $(x-2)/3=(y-3)/4=(z-4)/5$. ...(2)

Let l, m, n be the d.c.'s of the line of S.D. The line of S.D. being perpendicular to both the lines (1) and (2), we have

$2l+3m+4n=0$ and $3l+4m+5n=0$.

Solving these equations, we have

$$\frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

or $$\frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(-1)^2+(2)^2+(-1)^2\}}}=\frac{1}{\sqrt{6}}.$$ ...(3)

Clearly the line (1) passes through the point $A(1, 2, 3)$ and the line (2) passes through $B(2, 3, 4)$.

$\therefore$ The length of S.D. = The projection of join of A and B on the line of S.D. whose d.c.'s are l, m, n

$$=l(2-1)+m(3-2)+n(4-3)$$

$$=\frac{-1}{\sqrt{6}}(1)+\frac{2}{\sqrt{6}}(1)-\frac{1}{\sqrt{6}}(1)=0.$$

Since the length of S.D. = 0, hence the given lines are coplanar *i.e.* intersecting.

Ex. 5. *Find the points on the lines*

$$\frac{x-6}{3}=-(y-7)=z-4 \text{ and } \frac{-x}{3}=\frac{y+9}{2}=\frac{z-2}{4}$$

which are nearest to each other. Hence find the shortest distance between the lines and also its equations. **(Bundelkhand 1978)**

Sol. The equations of the given lines are

$$(x-6)/3=(y-7)/(-1)=(z-4)/1=r_1 \text{ (say)} \quad \ldots(1)$$

and
$$x/(-3)=(y+9)/2=(z-2)/4=r_2 \text{ (say)} \quad \ldots(2)$$

The points on the lines (1) and (2) which are nearest to each other are the points where the line of S.D. meets the lines (1) and (2).

Any point P on (1) is $(3r_1+6, -r_1+7, r_1+4)$ ...(3)
and any point Q on (2) is $(-3r_2, 2r_2-9, 4r_2+2)$. ...(4)

The d.r.'s of PQ are

$$-3r_2-3r_1-6,\ 2r_2+r_1-16,\ 4r_2-r_1-2. \quad \ldots(5)$$

Let the required points be P and Q, so that PQ is the line of S.D. Hence PQ is perpendicular to both the given lines (1) and (2) and so we have

$$3(-3r_2-3r_1-6)-1(2r_2+r_1-16)+1.(4r_2-r_1-2)=0$$

and $-3(-3r_2-3r_1-6)+2(2r_2+r_1-16)+4.(4r_2-r_1-2)=0$

or $-7r_2-11r_1-4=0$ and $29r_2+7r_1-22=0$.

Solving these equations we get $r_1=-1$, $r_2=1$.

Substituting these values in (3), (4) and (5), we have

$$P(3, 8, 3),\ Q(-3, -7, 6)$$

and d.r.'s of PQ (the line of S.D.) are $-6, -15, 3$ or $2, 5, -1$.

The length of S.D. $=PQ=\sqrt{\{(-3-3)^2+(-7-8)^2+(6-3)^2\}}$
$=\sqrt{\{(6)^2+(15)^2+(3)^2\}}=3\sqrt{(30)}$.

The line PQ of shortest distance is a line passing through $P(3, 8, 3)$ and having d.r.'s $2, 5, -1$ and hence its equations are

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}.$$

Ex. 6. *Show that the shortest distance between the lines* $x+a=2y=-12z$ *and* $x=y+2a=6z-6a$ *is* $2a$. **(Barahampur 1981)**

Sol. The equations of the given lines are

$$(x+a)/12=y/6=z/(-1), \quad \ldots(1)$$

and
$$x/6=(y+2a)/6=(z-a)/1. \quad \ldots(2)$$

Let l, m, n be the d.c.'s of the line of S.D. The line of S.D. being perpendicular to both the lines (1) and (2), we have

$$12l+6m-n=0 \text{ and } 6l+6m+n=0.$$

Solving these relations, we have

$$\frac{l}{6+6}=\frac{m}{-6-12}=\frac{n}{72-36}$$

or
$$\frac{l}{2}=\frac{m}{-3}=\frac{n}{6}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(2)^2+(-3)^2+(6)^2\}}}=\frac{1}{7}.$$

Clearly the line (1) passes through the point $A(-a, 0, 0)$ and the line (2) passes through $B(0, -2a, a)$.

$\therefore$ The length of S.D. = The projection of join of A and B on the line of S.D. whose d.c.'s are l, m, n.

$$=l(0+a)+m(-2a-0)+n(a-0)$$
$$=(2/7)\,a-(3/7)\,(-2a)+(6/7)\,a=2a.$$

Proved.

Ex. 7. *If the axes are rectangular, find the shortest distance between the lines* $y=az+b$, $z=\alpha x+\beta$, *and* $y=a'z+b'$, $z=\alpha' x+\beta'$. *Hence deduce the condition for the lines to be coplanar.*

Sol. The equations of the given lines in symmetrical form are given by

$$\frac{x+\beta/\alpha}{1/\alpha}=\frac{y-b}{a}=\frac{z}{1}; \quad \ldots(1)$$

and
$$\frac{x+\beta'/\alpha'}{1/\alpha'}=\frac{y-b'}{a'}=\frac{z}{1}. \quad \ldots(2)$$

Let l, m, n be the d.c.'s of the line of S.D. The line of S.D. being perpendicular to both the lines (1) and (2), we have

$$l.(1/\alpha)+m.a+n.1=0 \text{ and } l(1/\alpha')+ma'+n.1=0.$$

Solving these relations, we get

$$\frac{l}{a-a'}=\frac{m}{(1/\alpha')-(1/\alpha)}=\frac{n}{(1/\alpha)\,a'-(1/\alpha').a}$$

or
$$\frac{l}{\alpha\alpha'(a-a')}=\frac{m}{\alpha-\alpha'}=\frac{n}{a'\alpha'-a\alpha}$$

$$=\frac{1}{\sqrt{\{\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a'\alpha'-a\alpha)^2\}}} \quad \ldots(3)$$

Clearly the line (1) passes through the point $A(-\beta/\alpha, b, 0)$ and the line (2) passes through $B(-\beta'/\alpha', b', 0)$.

The length of S.D. = The projection of the join of A and B on the line of S.D. whose d.c.'s l, m, n are given by (3)

$$=l(-\beta'/\alpha'+\beta/\alpha)+m(b'-b)+n(0-0)$$
$$=\left(\frac{\alpha'\beta-\alpha\beta'}{\alpha\alpha'}\right)l+(b'-b)\,m$$
$$=\frac{(\alpha'\beta-\alpha\beta')\,\alpha\alpha'\,(a-a')}{\alpha\alpha'\sqrt{\{\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a'\alpha'-a\alpha)^2\}}}$$
$$+\frac{(b'-b)\,(\alpha-\alpha')}{\sqrt{\{\alpha^2\alpha'^2\,(a-a')^2+(\alpha-\alpha')^2+(a'\alpha'-a\alpha)^2\}}}$$

[Putting the values of l, m from (3)]

$$=\frac{(\alpha\beta'-\alpha'\beta)(a-a')+(b-b')(\alpha-\alpha')}{\sqrt{\{\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a\alpha-a'\alpha')^2\}}},$$

neglecting the negative sign.

This is the required S.D.

If the given lines are coplanar *i.e.* intersecting, then S.D.$=0$. Therefore the required condition is given by

$$(\alpha\beta'-\alpha'\beta)(a-a')+(b-b')(\alpha-\alpha')=0.$$

Ex. 8. *Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=a$ is $2a/\sqrt{6}$ and that the three lines of shortest distance intersect at the point $x=y=z=-a$.*

(Gorakhpur 1975; Lucknow 76, Meerut 87)

Sol. The equations of the edge of the tetrahedron determined by the planes $y+z=0$, $z+x=0$ are

$$x/1=y/1=z/(-1). \qquad \text{...(1)}$$

The equations of the edge opposite to that given by (1) are determined by the planes $x+y=0$, $x+y+z=a$ and hence are given by

$$x/1=y/(-1)=(z-a)/0. \qquad \text{...(2)}$$

Let l, m, n be the d.c.'s of the line of S.D. between (1) and (2). This line of S.D. being perpendicular to both the lines (1) and (2), we have

$$l.1+m.1+n.(-1)=0 \text{ and } l.1+m.(-1)+n.0=0.$$

Solving, $$\frac{l}{1}=\frac{m}{1}=\frac{n}{2}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{\{(1)^2+(1)^2+(2)^2\}}}=\frac{1}{\sqrt{6}}. \qquad \text{...(3)}$$

Clearly the line (1) passes through the point $A(0, 0, 0)$ and the line (2) passes through $B(0, 0, a)$.

$\therefore$ The length of S.D. = The projection of the join of A and B on the line of S.D. whose d.c.'s l, m, n are given by (3).

$$=l(0-0)+m(0-0)+n(a-0)$$
$$=na=2a/\sqrt{6}.$$

The equations of S.D. (See § 2).

The equation of the plane through the line (1) and S.D. is

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix}=0$$

or $$x(2+1)-y(2+1)+z(1-1)=0$$

or $$x-y=0. \qquad \text{...(4)}$$

The equation of the plane through the line (2) and S.D. is

$$\begin{vmatrix} x & y & z-a \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix}=0$$

or $$x(-2-0)-y(2-0)+(z-a)(1+1)=0$$

or $$x+y-z+a=0. \qquad \text{...(5)}$$

The equations (4) and (5) together are the equations of the S.D. These equations are clearly satisfied by the point

$$x=y=z=-a.$$

By the symmetry of the co-ordinates of the point

$$x=y=z=-a,$$

it follows that the other lines of S.D. between other two pairs of opposite edges will pass through the point $x=y=z=-a$.

Hence the three lines of shortest distance intersect at the point

$$x=y=z=-a.$$

Ex. 9. *Find the length and position of the S.D. between the lines*

$$\frac{x}{4}=\frac{y+1}{3}=\frac{z-2}{2},\ 5x-2y-3z+6=0=x-3y+2z-3.$$

Sol. The equations of the given lines are

$$x/4=(y+1)/3=(z-2)/2 \qquad \text{...(1)}$$

and $$5x-2y-3z+6=0=x-3y+2z-3. \qquad \text{...(2)}$$

Here we shall use method III of § 2.

The equation of any plane through the line (2) is

$$(5x-2y-3z+6)+\lambda(x-3y+2z-3)=0$$

or $$x(5+\lambda)+y(-2-3\lambda)+z(-3+2\lambda)+(6-3\lambda)=0. \qquad \text{...(3)}$$

If the plane (3) is parallel to the line (1), then the normal to the plane (3) will be perpendicular to the line (1) and hence we have

$$4(5+\lambda)+3(-2-3\lambda)+2(-3+2\lambda)=0 \quad \text{or} \quad \lambda=8.$$

Putting this value of λ in (3), the equation of the plane through the line (2) and parallel to the line (1) is given by

$$13x-26y+13z-18=0. \qquad \text{...(4)}$$

Clearly the line (1) is passing through the point $A(0, -1, 1)$.

$\therefore$ Length of S.D. = The length of perpendicular from the point $A(0, -1, 2)$ to the plane (4)

$$=\frac{13.0-26.(-1)+13.(2)-18}{\sqrt{\{(13)^2+(-26)^2+(13)^2\}}}$$

$$=\frac{34}{13\sqrt{6}}=\frac{34\sqrt{6}}{13\times 6}=\frac{17\sqrt{6}}{39}.$$

The equations of S.D. The equation of the plane through the line (1) and perpendicular to the plane (4) is given by

$$\begin{vmatrix} x & y+1 & z-2 \\ 4 & 3 & 2 \\ 13 & -26 & 13 \end{vmatrix}=0,$$

or $$13\begin{vmatrix} x & y+1 & z-2 \\ 4 & 3 & 2 \\ 1 & -2 & 1 \end{vmatrix}=0$$

or $x(3+4)-(y+1)(4-2)+(z-2)(-8-3)=0$

or $7x-2y-11z+20=0.$ ...(5)

Again the equation of any plane through the line (2) is

$(5x-2y-3z+6)+\mu(x-3y+2z-3)=0$

or $x(5+\mu)+y(-2-3\mu)+z(-3+2\mu)+(6-3\mu)=0.$ ...(6)

If (6) is perpendicular to the plane (4), then we have

$13(5+\mu)-26(-2-3\mu)+13(-3+2\mu)=0.$

Dividing it by 13, we get

$5+\mu-2(-2-3\mu)+(-3+2\mu)=0$, or $\mu=-2/3.$

Putting the value of μ in (6), the equation of the plane through the line (2) and perpendicular to the plane (4) is given by

$13x-13z+24=0.$ ...(7)

$\therefore$ The equation (5) and (7) together are the required equation of the S.D.

Ex. 10. *Find the length of the shortest distance between the z-axis and the line*

$x+y+2z-3=0=2x+3y+4z-4.$ **(Gorakhpur 1981)**

Sol. Here we shall use Method III of § 2.

The equations of z-axis are

$x/0=y/0=z/1.$ ...(1)

The equations of the other line are

$x+y+2z-3=0,\ 2x+3y+4z-4=0.$ ...(2)

The equation of any plane through the line (2) is

$(x+y+2z-3)+\lambda(2x+3y+4z-4)=0$

or $x(1+2\lambda)+y(1+3\lambda)+z(2+4\lambda)-3-4\lambda=0.$ ...(3)

If the plane (3) is parallel to the line (1) [*i.e.* z-axis] then the normal to the plane (3) will be perpendicular to the line (1) and hence we have

$0.(1+2\lambda)+0.(1+3\lambda)+1.(2+4\lambda)=0$, or $\lambda=-\frac{1}{2}.$

Putting this value of λ in (3), the equation of the plane through the line (2) and parallel to z-axis [*i.e.* (1)] is given by

$x\ 0+y.(1-3/2)+z.0-3+2=0$, or $y+2=0.$ ...(4)

Clearly z-axis *i.e.*, the line (1) passes through the point (0, 0, 0).

$\therefore$ Length of S.D. = the length of perpendicular from the point (0, 0, 0) to the plane (4)

$$=\frac{0+2}{\sqrt{\{(0)^2+(0)^2+(1)^2\}}}=\frac{2}{1}=2.$$

Ex. 11. *Find the shortest distance between the z-axis and the line*

$ax+by+cz+d=0=a'x+b'y+c'z+d'.$ **(Meerut 1984)**

Show also that it meets the z-axis at a point whose distance from the origin is

$$\frac{(ab'-d'b)(bc'-b'c)+(ca'-c'a)(ad'-a'd)}{(bc'-b'c)^2+(ca'-c'a)^2}.$$

(Agra 1979; Allahabad 78; Bundelkhand 79; Kanpur 75, 76, 79, 82; Lucknow 77, 82; Rajasthan 78)

Sol. The equations of the z-axis are

$\frac{x}{0}=\frac{y}{0}=\frac{z}{1}=r_1$ (say). ...(1)

The equations of the other line are

$ax+by+cz+d=0=a'x+b'y+c'z+d'.$ ...(2)

The length of S.D. We shall find it by Method III.

The equation of any plane through the line (2) is

$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$

or $x(a+\lambda a')+y(b+\lambda b')+z(c+\lambda c')+(d+\lambda d')=0.$ ...(3)

If the plane (3) is parallel to the line (1) [*i.e.* z-axis], then

$0(a+\lambda a')+0.(b+\lambda b')+1.(c+\lambda c')=0$ or $\lambda=-c/c'$

Putting this value of λ in (3), the equation of the plane through the line (2) and parallel to z-axis is given by

$x(a-ca'/c')+y(b-cb'/c')+z(c-c)+(d-cd'/c')=0$

or $x(ac'-ca')+y(bc'-cb')+(dc'-cd')=0.$ ...(4)

Clearly z-axis *i.e.*, the line (1) passes through the point (0, 0, 0).

$\therefore$ Length of S.D. = the length of perpendicular from the point (0, 0, 0) to the plane (4)

$$=\frac{0+0+(dc'-cd')}{\sqrt{\{(ac'-ca')^2+(bc'-b'c)^2\}}}$$

$$=\frac{dc'-d'c}{\sqrt{\{(ac'-a'c)^2+(bc'-b'c)^2\}}}.\qquad\textbf{Proved.}$$

The distance from origin of the point where the line of S.D. meets the z-axis. It will be convenient to use method II.

The equations of line (2) in symmetrical form are

$$\frac{x-\dfrac{bd'-b'd}{ab'-a'b}}{bc'-b'c}=\frac{y-\dfrac{a'd-ad'}{ab'-a'b}}{ca'-c'a}=\frac{z-0}{ab'-a'b}\qquad\ldots(5)$$

[See § 4, Chapter 4]

For convenience, let (5) be written as

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-0}{n_2}=r_2\text{ (say)}.\qquad\ldots(6)$$

Since 0, 0, 1 are the actual d.c.'s of the z-axis [*i.e.* the line (1)], therefore r_1 is the actual distance of any point on the line (1) from (0, 0, 0). Let $P(0, 0, r_1)$ be any point on the line (1).

Also any point on the line (6) is $(l_2r_2+x_2,\ m_2r_2+y_2,\ n_2r_2)$, say the point Q.

The d.r.'s of PQ are

$$l_2r_2+x_2,\ m_2r_2+y_2,\ n_2r_2-r_1.$$

Let the line PQ be the line of S.D., so that PQ is perpendicular to both the lines (1) and (2), and therefore, we have

$$0.(l_2r_2+x_2)+0.(m_2r_2+y_2)+1.(n_2r_2-r_1)=0\quad\text{or}\quad r_1=n_2r_2$$

and $\quad l_2(l_2r_2+x_2)+m_2(m_2r_2+y_2)+n_2(n_2r_2-r_1)=0$

or $\quad l_2^2r_2+l_2x_2+m_2^2r_2+m_2y_2+0=0\qquad[\because\ r_1=n_2r_2]$

or $\quad r_2(l_2^2+m_2^2)=-(l_2x_2+m_2y_2)$

or $\quad r_2=-(l_2x_2+m_2y_2)/(l_2^2+m_2^2).$

$\therefore\quad r_1=n_2r_2=-n_2(l_2x_2+m_2y_2)/(l_2^2+m_2^2)$

Substituting the corresponding values of l_2, m_2, n_2, x_2 and y_2 from (5) and (6), we have

$$r_1=\frac{-(ab'-a'b)}{(bc'-b'c)^2+(ca'-c'a)^2}\left\{(bc'-b'c).\frac{(bd'-b'd)}{ab'-a'b}+(ca'-c'a).\frac{a'd-ad'}{(ab'-a'b)}\right\}$$

$$=\frac{(db'-d'b)(bc'-b'c)+(ca'-c'a)(ad'-a'd)}{(bc'-b'c)^2+(ca'-c'a)^2}.\qquad\textbf{Proved.}$$

Ex. 12. *Show that the shortest distance between the lines*

$$\frac{x-x_1}{\cos\alpha_1}=\frac{y-y_1}{\cos\beta_1}=\frac{z-z_1}{\cos\gamma_1},\ \frac{x-x_2}{\cos\alpha_2}=\frac{y-y_2}{\cos\beta_2}=\frac{z-z_2}{\cos\gamma_2}$$

meets the first line in a point whose distance from (x_1, y_1, z_1) *is*

$$[\Sigma\{(x_1-x_2)(\cos\alpha_1-\cos\theta\cos\alpha_2)\}]/\sin^2\theta$$

where θ *is the angle between the lines.*

Sol. The epuations of the given lines are

$$\frac{x-x_1}{\cos\alpha_1}=\frac{y-y_1}{\cos\beta_1}=\frac{z-z_1}{\cos\gamma_1}=r_1\text{ (say)}\qquad\ldots(1)$$

and

$$\frac{x-x_2}{\cos\alpha_2}=\frac{y-y_2}{\cos\beta_2}=\frac{z-z_2}{\cos\gamma_2}=r_2\text{ (say)}.\qquad\ldots(2)$$

Any point P on (1) is

$$(r_1\cos\alpha_1+x_1,\ r_1\cos\beta_1+y_1,\ r_1\cos\gamma_1+z_1)$$

and any point Q on (2) is

$$(r_2\cos\alpha_2+x_2,\ r_2\cos\beta_2+y_2,\ r_2\cos\gamma_2+z_2).$$

Let the shortest distance meet the line (1) at P and the line (2) at Q.

Now, since $\cos\alpha_1, \cos\beta_1, \cos\gamma_1$ are the actual d.c.'s of the line (1), therefore r_1 is the actual distance of P from the point (x_1, y_1, z_1) on the line (1). It is required to find r_1.

D.r.'s of PQ are $r_1\cos\alpha_1+x_1-r_2\cos\alpha_2-x_2,\ r_1\cos\beta_1+y_1-r_2\cos\beta_2-y_2,\ r_1\cos\gamma_1+z_1-r_2\cos\gamma_2-z_2.$

Since PQ is the line of shortest distance, therefore, it is perpendicular to both the lines (1) and (2), and hence, we have

$$\cos\alpha_1(r_1\cos\alpha_1+x_1-r_2\cos\alpha_2-x_2)+\cos\beta_1(r_1\cos\beta_1+y_1-r_2\cos\beta_2-y_2)+\cos\gamma_1(r_1\cos\gamma_1+z_1-r_2\cos\gamma_2-z_2)=0\qquad\ldots(3)$$

and

$$\cos\alpha_2(r_1\cos\alpha_1+x_1-r_2\cos\alpha_2-x_2)+\cos\beta_2(r_1\cos\beta_1+y_1-r_2\cos\beta_2-y_2)+\cos\gamma_2(r_1\cos\gamma_1+z_1-r_2\cos\gamma_2-z_2)=0.\qquad\ldots(4)$$

Now

$$\left.\begin{array}{l}\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1=1,\ \cos^2\alpha_2+\cos^2\beta_2+\cos^2\gamma_2=1\\ \text{and }\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=\cos\theta,\end{array}\right\}\qquad\ldots(5)$$

θ being given to be the angle between the lines (1) and (2).

From equation (3), we have

$$r_1(\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1)-r_2(\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2)+(x_1-x_2)\cos\alpha_1+(y_1-y_2)\cos\beta_1+(z_1-z_2)\cos\gamma_1=0$$

or $\quad r_1 - r_2 \cos\theta + \Sigma\{(x_1-x_2)\cos\alpha_1\}=0 \qquad \{\text{using }(5)\}. \qquad \ldots(6)$

Similarly equation (4) may be written as

$$-r_2 + r_1\cos\theta + \Sigma\{(x_1-x_2)\cos\alpha_2\}=0. \qquad \ldots(7)$$

Multiplying (7) by $\cos\theta$ and subtracting from (6), we have

$$r_1(1-\cos^2\theta)+\Sigma\{(x_1-x_2)\cos\alpha_1\}-\Sigma\{(x_1-x_2)\cos\alpha_2\}\cos\theta=0$$

or $\quad r_1\sin^2\theta+\Sigma\{(x_1-x_2)(\cos\alpha_1-\cos\alpha_2\cos\theta)\}=0$

or $\quad r_1=[\Sigma\{(x_1-x_2)(\cos\alpha_1-\cos\alpha_2\cos\theta)\}]/\sin^2\theta,$

neglecting the negative sign because r_1 is the distance.

Ex. 13. *Show that the equation of the plane containing the line $y/b+z/c=1$, $x=0$ and parallel to the line*

$$x/a-z/c=1,\ y=0 \text{ is } x/a-y/b-z/c+1=0$$

and if 2d is the shortest distance then show that $d^{-2}=a^{-2}+b^{-2}+c^{-2}$.

(Agra 1977, 81; Kanpur 82; Punjab 82; Ranchi 74; Meerut 87P, 90P

Sol. The equations of the given lines are

$$y/b+z/c=1,\ x=0 \qquad \ldots(1)$$

and $\quad \dfrac{x}{a}-\dfrac{z}{c}=1,\ y=0 \quad \text{or} \quad \dfrac{x-a}{a}=\dfrac{y}{0}=\dfrac{z}{c}, \qquad \ldots(2)$

the equations of the second line being put in symmetrical form

The equation of any plane through the line (1) is

$$(y/b+z/c-1)+\lambda x=0 \text{ or } \lambda x+(1/b)\,y+(1/c)\,z-1=0. \qquad \ldots(3)$$

If the plane (3) is parallel to the line (2), then the normal to the plane (3) whose d.r.'s are λ, $1/b$, $1/c$ will be perpendicular to the line (2), and so we have

$$a.\lambda+0.(1/b)+c.(1/c)=0 \quad\text{or}\quad \lambda=-(1/a).$$

Putting this value of λ in (3), the equation of the plane containing the line (1) and parallel to the line (2) is

$$-\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0 \quad\text{or}\quad \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0. \qquad \ldots(4)$$

Clearly $(a, 0, 0)$ is a point on the line (2). Hence the length $2d$ of the S.D. = the perpendicular distance of $(a, 0, 0)$ from the plane (4).

$$\therefore\quad 2d=\frac{a(1/a)-0-0+1}{\sqrt{\{(1/a)^2+(-1/b)^2+(-1/c)^2\}}}=\frac{2}{\sqrt{(a^{-2}+b^{-2}+c^{-2})}}$$

or $\quad d^2=1/(a^{-2}+b^{-2}+c^{-2})$ or $d^{-2}=a^{-2}+b^{-2}+c^{-2}$.

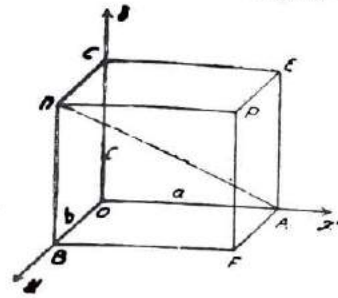
Ex. 14. *Show that the shortest distance between the diagonals of a rectangular parallelopiped and the edges not meeting it are*

$$bc/\sqrt{(b^2+c^2)},\ ca/\sqrt{(c^2+a^2)},\ ab/\sqrt{(a^2+b^2)}$$

where a, b, c are the lengths of the edges.

(Punjab 1981; Garhwal 78S)

Sol. Consider a rectangular parallelopiped whose three coterminous edges OA, OB, OC are taken along the axes of x, y and



z respectively. Also $OA=a$, $OB=b$ and $OC=c$. The co-ordinates of the different vertices are as follows ;

$O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$, $D(0, b, c)$, $E(a, 0, c)$, $F(a, b, 0)$, $P(a, b, c)$.

Consider a diagonal AD and the edge OB not meeting this diagonal. Now we shall find the length of S. D. between AD and OB.

The equations of AD are

$$\frac{x-a}{0-a}=\frac{y-0}{b-0}=\frac{z-0}{c-0} \quad\text{or}\quad \frac{x-a}{-a}=\frac{y}{b}=\frac{z}{c}. \qquad \ldots(1)$$

The equations of OB are

$$\frac{x}{0}=\frac{y-b}{1}=\frac{z}{0}. \qquad \ldots(2)$$

Let l, m, n be the d.c.'s of the line of S.D. between the lines (1) and (2). Since the line of S D. is perpendicular to both the lines (1) and (2), we have

$$l(-a)+m.b+n.c=0 \text{ and } l.0+m.1+n.0=0.$$

Solving, $\quad \dfrac{l}{c}=\dfrac{m}{0}=\dfrac{n}{a}=\dfrac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(c^2+0+a^2)}}=\dfrac{1}{\sqrt{(c^2+a^2)}}. \qquad \ldots(3)$

$\therefore$ The length of S.D. between AD and OB

= the projection of the join of $(a, 0, 0)$ {a point on (1)} and $(0, b, 0)$ {a point on (2)} on the line of S.D. whose d.c.'s l, m, n are given by (3)

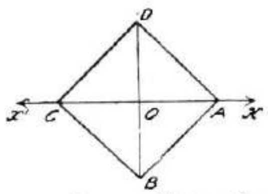
$= l\,(a-0)+m\,(0-b)+n\,(0-0)=la$

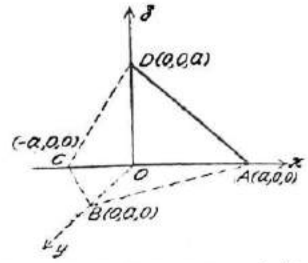
$= ac/\sqrt{(c^2+a^2)}. \qquad \ldots(4)$

Similarly the shortest distance between other pairs of lines can be found.

Ex. 15. *A square ABCD of diagonal 2a is folded along the diagonal AC so that the planes DAC, BAC are at right angles. Find the shortest distance between DC and AB.* **(Agra 1976, 80; M.U. 90)**

Sol. $ABCD$ is a square of diagonal $2a$, so that $AC=BD=2a$. Let O, the centre of the square, be chosen as origin of co-ordinates and the diagonal CA be taken along x-axis. Hence the





co-ordinates of the vertices A and C are $(a, 0, 0)$ and $(-a, 0, 0)$ respectively.

Now as given in the problem, the square is folded over along the diagonal AC so that the planes DAC and BAC are at right angles. This implies that the lines OB and OD become at right angles. Also OA is perpendicular to the plane DOB. Hence the lines OA, OB, OD are mutually orthogonal. Let us now take OB and OD as y and z axes respectively.

$\therefore$ The co-ordinates of B and D are $(0, a, 0)$ and $(0, 0, a)$ respectively.

The equations to AB are $\dfrac{x-a}{a}=\dfrac{y-0}{-a}=\dfrac{z-0}{0}$ ...(1)

The equation to DC are $\dfrac{x-0}{a}=\dfrac{y-0}{0}=\dfrac{z-a}{a}$ ...(2)

The equation of any plane through DC and parallel to AB [*i.e.* through the line (2) and parallel to the line (1)] is

$$\begin{vmatrix} x-0 & y-0 & z-a \\ a & 0 & a \\ a & -a & 0 \end{vmatrix}=0$$

or $$x(a^2)-y(-a^2)+(z-a)(-a^2)=0$$

or $$x+y-z+a=0. \quad \text{...(3)}$$

$\therefore$ The S.D. between DC and AB

= the length of perpendicular from a point $(a, 0, 0)$ on AB [*i.e.* (1)] to the plane (3)

$$=\frac{a+0-0+a}{\sqrt{\{(1)^2+(1)^2+(-1)^2\}}}=\frac{2a}{\sqrt{3}}.$$

Ex. 16. *Find the length and equations of the shortest distance*

between $\quad 3x-9y+5z=0=x+y-z$

and $\quad 6x+8y+3z-13=0=x+2y+z-3.$

(Meerut 1984 P, 86)

Sol. Here we shall use method IV of § 2. The equations of the planes through the given lines are

$$(3x-9y+5z)+\lambda_1(x+y-z)=0$$

and $$(6x+8y+3z-13)+\lambda_2(x+2y+z-3)=0$$

or $$x(3+\lambda_1)+y(-9+\lambda_1)+z(5-\lambda_1)=0 \quad \text{...(1)}$$

and $$x(6+\lambda_2)+y(8+2\lambda_2)+z(3+\lambda_2)-(13+3\lambda_2)=0. \quad \text{...(2)}$$

If the planes (1) and (2) are parallel, then their coefficients are proportional and so we have

$$\frac{3+\lambda_1}{6+\lambda_2}=\frac{-9+\lambda_1}{8+2\lambda_2}=\frac{5-\lambda_1}{3+\lambda_2}=k \text{ (say)}. \quad \text{...(3)}$$

Taking the ratios 1st, 2nd and 3rd with k respectively in (3), we get

$(3+\lambda_1)=k(6+\lambda_2)$ or $3+\lambda_1-6k-k\lambda_2=0$ ...(4)

$(-9+\lambda_1)=k(8+2\lambda_2)$ or $-9+\lambda_1-8k-2k\lambda_2=0$ ...(5)

$(5-\lambda_1)=k(3+\lambda_2)$ or $5-\lambda_1-3k-k\lambda_2=0.$ ...(6)

Substituting (6) from (4),

$$-2+2\lambda_1-3k=0. \quad \text{...(7)}$$

Subtracting 2 times (6) from (5),

$$-19+3\lambda_1-2k=0. \quad \text{...(8)}$$

Solving (7) and (8), $\lambda=53/5$, $k=32/5$.

Putting the values of λ_1 and k in (4), we get $\lambda_2=-31/8$.

Substituting the values of λ_1 and λ_2 in (1) and (2), the equations of the parallel planes through the given lines are

$$17x+2y-7z=0 \quad \text{...(9)}$$

and $$17x+2y-7z-11=0. \quad \text{...(10)}$$

The required S.D. is the distance between the parallel planes (9) and (10).

Any point on the plane (9) is (0, 0, 0).

$\therefore$ The length of S.D. = the length of perpendicular from (0, 0, 0) to the plane (10)

$$=\frac{0+0-0-11}{\sqrt{\{(17)^2+(2)^2+(-7)^2\}}}=\frac{11}{\sqrt{(342)}}$$ [Numerically]

The equations of S.D.

The equation of any plane through the first given line is

$$x\,(3+\lambda_1)+y\,(-9+\lambda_1)+z\,(5-\lambda_1)=0 \text{ [See (1)]} \quad ...(11)$$

If the plane (11) is perpendicular to (9) or (10), we have

$$17\,(3+\lambda_1)+2\,(-9+\lambda_1)-7\,(5-\lambda_1)=0 \text{ or } \lambda_1=1/13.$$

Putting the value of λ_1 in (11) the equation of the plane through the 1st given line and perpendicular to the plane (9) or (10) is given by

$$10x-29y+16z=0. \quad ...(12)$$

Again the equation of any plane through the 2nd given line is [See equation (2)]

$$x\,(6+\lambda_2)+y\,(8+2\lambda_2)+z\,(3+\lambda_2)-(13+3\lambda_2)=0. \quad ...(13)$$

If the plane (3) is perpendicular to (9) or (10), we have

$$17\,(6+\lambda_2)+2\,(8+2\lambda_2)-7\,(3+\lambda_2)=0 \text{ or } \lambda_2=-58/7.$$

Putting the value of λ_2 in (13), the equation of the plane through the 2nd given line and perpendicular to the plane (9) or (10) is given by

$$13x+82y+55z-109=0. \quad ...(14)$$

The equations (12) and (14) are the required equations of the shortest distance.

Note. We can solve the above problem by reducing both the lines to symmetrical form and then using method I or II. The problem can also be solved by reducing only one line to symmetrical form and then using method III.

Ex. 17. *Find the shortest distance between the lines*

$$x=0,\ \tfrac{1}{2}y+\tfrac{1}{3}z=1$$

and $y=0,\ \frac{1}{4}x-\frac{1}{3}z=1.$ **(Agra 1982)**

Sol. The equations of the 1st line in symmetrical form are

$$\frac{x}{0}=\frac{y-2}{2}=\frac{z}{-3}. \quad ...(1)$$

The equations of the 2nd line are

$$y=0,\ \tfrac{1}{4}x-\tfrac{1}{3}z=1 \quad ...(2)$$

The equation of any plane through the line (2) is

$$(\tfrac{1}{4}x-\tfrac{1}{3}z-1)+\lambda y=0$$

or $$3x+12\lambda y-4z-12=0. \quad ...(3)$$

If the plane (3) is parallel to the line (1), then the normal to (3) is perpendicular to the line (1), and hence we have

$$0.3+2.12\lambda-3.(-4)=0 \text{ or } \lambda=-\tfrac{1}{2}.$$

Putting the value of λ in (3), the equation of the plane through the line (2) and parallel to the line (1) is given by

$$3x-6y-4z-12=0. \quad (4)$$

The line (1) clearly passes through the point (0, 2, 0).

The length of S.D. = the perpendicular distance of (0, 2, 0) from the plane (4)

$$=\frac{0-12-0-12}{\sqrt{\{(3)^2+(-6)^2+(-4)^2\}}}=\frac{24}{\sqrt{(61)}} \text{ (Numerically).}$$

Ex. 18. *Prove that the S.D. between the lines*

$$ax+by+cz+d=0=a'x+b'y+c'z+d'$$

and $$\alpha x+\beta y+\gamma z+\delta=0=\alpha' x+\beta' y+\gamma' z+\delta' \text{ is}$$

$$\begin{vmatrix} d & d' & \delta & \delta' \\ a & a' & \alpha & \alpha' \\ b & b' & \beta & \beta' \\ c & c' & \gamma & \gamma' \end{vmatrix} \div \sqrt{[\Sigma\,(BC'-B'C)^2]}$$

where $A=bc'-b'c$ *and* $A'=\beta\gamma'-\beta'\gamma$ *etc.*

Sol. The equations of the given lines are

$$ax+by+cz+d=0=a'x+b'y+c'z+d' \quad ...(1)$$

and $$\alpha x+\beta y+\gamma z+\delta=0=\alpha' x+\beta' y+\gamma' z+\delta'. \quad ...(2)$$

We shall use method IV. The equations of any planes through the given lines (1) and (2) are

$$(ax+by+cz+d)+\lambda_1\,(a'x+b'y+c'z+d')=0 \quad ...(3)$$

or $$x\,(a+\lambda_1 a')+y\,(b+\lambda_1 b')+z\,(c+\lambda_1 c')+(d+\lambda_1 d')=0 \quad ...(3')$$

and $$(\alpha x+\beta y+\gamma z+\delta)+\lambda_2\,(\alpha' x+\beta' y+\gamma' z+\delta')=0 \quad ...(4)$$

or $$x\,(\alpha+\lambda_2\alpha')+y\,(\beta+\lambda_2\beta')+z\,(\gamma+\lambda_2\gamma')+(\delta+\lambda_2\delta')=0 \quad ...(4')$$

If the planes (3') and (4') [*i.e.* (3) and (4)] are parallel, then their corresponding coefficients are proportional and so we have

$$\frac{a+\lambda_1 a'}{\alpha+\lambda_2\alpha'}=\frac{b+\lambda_1 b'}{\beta+\lambda_2\beta'}=\frac{c+\lambda_1 c'}{\gamma+\lambda_2\gamma'}=k \text{ (say)}.$$

From these relations, we get

$$a+\lambda_1 a'-k\alpha-k\lambda_2\alpha'=0 \quad ...(5)$$

$$b+\lambda_1 b'-k\beta-k\lambda_2\beta'=0 \qquad ...(6)$$
$$c+\lambda_1 c'-k\gamma-k\lambda_2\gamma'=0. \qquad ...(7)$$

Eliminating λ_1, $-k$, $-k\lambda_2$ between (3), (5), (6) and (7), the equation of the plane through the line (1) and parallel to (2) is given by

$$\begin{vmatrix} ax+by+cz+d & a'x+b'y+c'z+d' & 0 & 0 \\ a & a' & \alpha & \alpha' \\ b & b' & \beta & \beta' \\ c & c' & \gamma & \gamma' \end{vmatrix}=0.$$

Adding $(-x)$ times second, $(-y)$ times third and $(-z)$ times fourth row to the first row, we get

$$\begin{vmatrix} d & d' & -(\alpha x+\beta y+\gamma z) & -(\alpha' x+\beta' y+\gamma' z) \\ a & a' & \alpha & \alpha' \\ b & b' & \beta & \beta' \\ c & c' & \gamma & \gamma' \end{vmatrix}=0 \qquad ...(8)$$

Now we shall evaluate the coefficients of x, y and z in the expansion of the determinant in (8).

The coefficient of x

$$=-\alpha\begin{vmatrix} a & a' & \alpha' \\ b & b' & \beta' \\ c & c' & \gamma' \end{vmatrix}-(-\alpha')\begin{vmatrix} a & a' & \alpha \\ b & b' & \beta \\ c & c' & \gamma \end{vmatrix}$$

$$=\begin{vmatrix} a & a' & -\alpha\alpha' \\ b & b' & -\alpha\beta' \\ c & c' & -\alpha\gamma' \end{vmatrix}+\begin{vmatrix} a & a' & \alpha\alpha' \\ b & b' & \beta\alpha' \\ c & c' & \gamma\alpha' \end{vmatrix}$$

$$=\begin{vmatrix} a & a' & \alpha\alpha'-\alpha\alpha' \\ b & b' & \beta\alpha'-\alpha\beta' \\ c & c' & \gamma\alpha'-\alpha\gamma' \end{vmatrix} \quad \begin{vmatrix} a & a' & 0 \\ b & b' & -C' \\ c & c' & B' \end{vmatrix}$$

$$=-C'(ac'-a'c)+B'(ab'-a'b)$$

[∵ It is given that $B'=\gamma\alpha'-\gamma'\alpha$, $C'=\alpha\beta'-\alpha'\beta$]

$$=C'(-B)+B'(C)=CB'-BC'.$$

Similarly the coefficient of $y=AC'-CA'$

and the coefficient of $z=BA'-AB'$. ...(9)

Now suppose (x_1, y_1, z_1) is a point on the line (2), so that we have

$$\alpha x_1+\beta y_1+\gamma z_1+\delta=0 \text{ and } \alpha' x_1+\beta' y_1+\gamma' z_1+\delta'=0$$

or $\alpha x_1+\beta y_1+\gamma z_1=-\delta$ and $\alpha' x_1+\beta' y_1+\gamma' z_1=-\delta'$ ...(10)

Now the required S.D. between the lines (1) and (2)

= The length of the perpendicular from the point (x_1, y_1, z_1) on the line (2) from the plane (8)

$$=\begin{vmatrix} d & d' & -(\alpha x_1+\beta y_1+\gamma z_1) & -(\alpha' x_1+\beta' y_1+\gamma' z_1) \\ a & a' & \alpha & \alpha' \\ b & b' & \beta & \beta' \\ c & c' & \gamma & \gamma' \end{vmatrix}$$

$\div\sqrt{[(\text{coefficient of } x)^2+(\text{coeff. of } y)^2+(\text{coeff. of } z)^2]}$.

Putting values from (9) and (10), we get the required S.D.

$$=\begin{vmatrix} d & d' & \delta & \delta' \\ a & a' & \alpha & \alpha' \\ b & b' & \beta & \beta' \\ c & c' & \gamma & \gamma' \end{vmatrix}\div\sqrt{[\Sigma(BC'-B'C)^2]}.$$

Proved.

Ex. 19. *Two straight lines*

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1};\ \frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$$

are cut by a third line whose direction cosines are λ, μ, ν. *Show that 'd' the length intercepted on the third line is given by*

$$d\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ \lambda & \mu & \nu \end{vmatrix}=\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}.$$

Deduce the length of the shortest distance between the first two lines.

Sol. The equations of the given lines are

$$(x-\alpha_1)/l_1=(y-\beta_1)/m_1=(z-\gamma_1)/n_1=r_1 \text{ (say)} \qquad ...(1)$$

and $(x-\alpha_2)/l_2=(y-\beta_2)/m_2=(z-\gamma_2)/n_2=r_2$ (say). ...(2)

Any point on the line (1) is $P\,(l_1r_1+\alpha_1,\ m_1r_1+\beta_1,\ n_1r_1+\gamma_1)$.

Any point on the line (2) is

$$Q\,(l_2r_2+\alpha_2,\ m_2r_2+\beta_2,\ n_2r_2+\gamma_2). \quad ...(3)$$

Let the third line with d.c.'s λ, μ, ν meet the line (1) at the point P and the line (2) at Q so that $PQ=d$.

Now the third line with d.c.'s λ, μ, ν is passing through the point P, hence its equations are

$$\frac{x-(l_1r_1+\alpha_1)}{\lambda}=\frac{y-(m_1r_1+\beta_1)}{\mu}=\frac{z-(n_1r_1+\gamma_1)}{\nu}=d \text{ (say)}. \quad ...(4)$$

$\therefore$ The co-ordinates of the point Q at a distance 'd' from the point P on the line (4) are

$$d\lambda+l_1r_1+\alpha_1,\ d\mu+m_1r_1+\beta_1,\ d\nu+n_1r_1+\gamma_1 \quad ...(5)$$

Now the co-ordinates given by (3) and (5) are of the same point Q. Hence comparing (3) and (5), we get

$$\left.\begin{aligned} d\lambda+l_1r_1+\alpha_1&=l_2r_2+\alpha_2\\ d\mu+m_1r_1+\beta_1&=m_2r_2+\beta_2\\ d\nu+n_1r_1+\gamma_1&=n_2r_2+\gamma_2\end{aligned}\right\}$$

or

$$\left.\begin{aligned} d\lambda+(\alpha_1-\alpha_2)+l_1r_1-l_2r_2&=0\\ d\mu+(\beta_1-\beta_2)+m_1r_1-m_2r_2&=0\\ d\nu+(\gamma_1-\gamma_2)+n_1r_1-n_2r_2&=0\end{aligned}\right\} \quad ...(6)$$

Eliminating r_1 and r_2 from the relations (6), we get

$$\begin{vmatrix} d\lambda+(\alpha_1-\alpha_2) & l_1 & l_2\\ d\mu+(\beta_1-\beta_2) & m_1 & m_2\\ d\nu+(\gamma_1-\gamma_2) & n_1 & n_2\end{vmatrix}=0.$$

Spliting this determinant into two determinants, we have

$$d\begin{vmatrix} \lambda & l_1 & l_2\\ \mu & m_1 & m_2\\ \nu & n_1 & n_2\end{vmatrix}+\begin{vmatrix} \alpha_1-\alpha_2 & l_1 & l_2\\ \beta_1-\beta_2 & m_1 & m_2\\ \gamma_1-\gamma_2 & n_1 & n_2\end{vmatrix}=0$$

or

$$d\begin{vmatrix} \lambda & l_1 & l_2\\ \mu & m_1 & m_2\\ \nu & n_1 & n_2\end{vmatrix}=-\begin{vmatrix} \alpha_1-\alpha_2 & l_1 & l_2\\ \beta_1-\beta_2 & m_1 & m_2\\ \gamma_1-\gamma_2 & n_1 & n_2\end{vmatrix}$$

or

$$d\begin{vmatrix} l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ \lambda & \mu & \nu\end{vmatrix}=-\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix} \quad ...(7)$$

Since d is the distance, hence neglecting the negative sign in (7), the required result is obtained.

Now if d stands for the S.D. between the given lines (1) and (2), then the third line with d.c.'s λ, μ, ν is perpendicular to both the given lines (1) and (2) and hence we have

$$l_1\lambda+m_1\mu+n_1\nu=0,\ l_2\lambda+m_2\mu+n_2\nu=0.$$

Solving, we get

$$\frac{\lambda}{(m_1n_2-m_2n_1)}=\frac{\mu}{n_1l_2-n_2l_1}=\frac{\nu}{l_1m_2-l_2m_1}=\frac{\sqrt{(\lambda^2+\mu^2+\nu^2)}}{\sqrt{[\Sigma\ (m_1n_2-m_2n_1)^2]}}$$

$$=\frac{1}{\sqrt{[\Sigma\ (m_1n_2-m_2n_1)^2]}}. \quad ...(8)$$

The value of the distance d given by (7) will become the length of S.D. between the given lines (1) and (2) if the values of λ, μ, ν are substituted in it from (8).

Now the coefficient of d in (7)

$$=\begin{vmatrix} l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ \lambda & \mu & \nu\end{vmatrix}$$

$$=\lambda\,(m_1n_2-m_2n_1)+\mu\,(n_1l_2-n_2l_1)+\nu\,(l_1m_2-l_2m_1)$$

$$=\frac{\Sigma\ (m_1n_2-m_2n_1)^2}{\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]}}$$

[putting the values of λ, μ, ν from (8)]

$$=\sqrt{[\Sigma\ (m_1n_2-m_2n_1)^2]}.$$

Using this value for the coefficient of d in (7), the S.D. 'd' is given by

$$d=\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix}\div\sqrt{[\Sigma\ (m_1n_2-m_2n_1)^2]}.$$

Exercises

1. Find the equations of the straight line perpendicular to both of the lines
$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+2}{2} \text{ and } \frac{x+2}{2}=\frac{y-5}{-1}=\frac{z+3}{2}.$$
Ans. $\dfrac{x-2}{7}=\dfrac{y-3}{4}=\dfrac{z-1}{-5}$.

2. Find the length and equations of the common perpendicular to the two lines

$$\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2} \text{ and } \frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}.$$

(Meerut 1984; Andhra 68; Gorakhpur 74; Madras 76)

Ans. The length of common perpendicular (*i.e.* S.D.)$=9$.
The equations of S.D. are
$32x+34y+13z-108=0=4x+11y+5z-27$.

3. Show that the length of shortest distance between the lines

$$\frac{x-2}{2}=\frac{y+1}{3}=\frac{z}{4};$$

$2x+3y-5z-6=0=3x-2y-z+3$ is $97/(13\sqrt{6})$.

(Rajasthan 1975)

4. Find the length and the equations of the shortest distance between the lines

$5x-y-z=0=x-2y+z+3$

and $7x-4y-2z=0=x-y+z-3$. **(Meerut 1986 S)**

Ans. The length of S. D. is $13/5\sqrt{2}$.
The equations of S. D. are
$17x+20y-19z-39=0,\ 8x+5y-31z+67=0$.

5. Find the equation of the shortest distance and its length between the lines

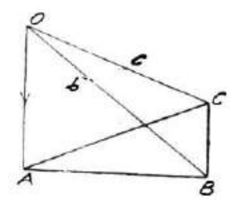
$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-2}{1};$$

$$\frac{x-1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}.$$

(M. U. 1990)

6
Volume of Tetrahedron

§ 1. (A) *To find the volume of a tetrahedron, whose three coterminous edges in the right-handed orientation are* **a, b, c**, *where* **a, b, c** *are vectors.*



Let $OABC$ be a tetrahedron. Let O be the origin and let position vectors of the vertices A, B, C be **a, b, c** respectively, so that

$\vec{OA}=\mathbf{a},\ \vec{OB}=\mathbf{b},\ \vec{OC}=\mathbf{c}$.

Then the volume V of the tetrahedron is given by

$V=\frac{1}{3}$ (area of the triangle OBC) $\times$ (perpendicular length from A on the plane OBC). ...(1)

Now the area of $\triangle OBC=\frac{1}{2}\,|\,\mathbf{b}\times\mathbf{c}\,|$. ...(2)

If $\hat{\mathbf{n}}$ be the unit vector perpendicular to the plane of the triangle OBC such that **b**, **c** and $\hat{\mathbf{n}}$ are in right handed orientation, then

$$\hat{\mathbf{n}}=\frac{\mathbf{b}\times\mathbf{c}}{|\,\mathbf{b}\times\mathbf{c}\,|},$$

since **b**, **c**, $\mathbf{b}\times\mathbf{c}$ are in right handed orientation.

$\therefore$ the length of the perpendicular from A on the plane OBC= the length of the projection of OA on the perpendicular to the plane OBC in the direction of $\hat{\mathbf{n}}$

$$=\vec{OA}\cdot\hat{\mathbf{n}}=\mathbf{a}\cdot\frac{\mathbf{b}\times\mathbf{c}}{|\,\mathbf{b}\times\mathbf{c}\,|}=\frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{|\,\mathbf{b}\times\mathbf{c}\,|}=\frac{[\mathbf{a},\mathbf{b},\mathbf{c}]}{|\,\mathbf{b}\times\mathbf{c}\,|}. \quad ...(3)$$

Putting the values from (2) and (3) in (1), we get

$$V=\tfrac{1}{2}\cdot\tfrac{1}{3}\,|\,\mathbf{b}\times\mathbf{c}\,|\,\frac{[\mathbf{a},\mathbf{b},\mathbf{c}]}{|\,\mathbf{b}\times\mathbf{c}\,|}$$

or

$$V=\tfrac{1}{6}\,[\mathbf{a},\mathbf{b},\mathbf{c}]. \quad ...(4)$$

This is the required formula for the volume of the tetrahedron.

(B) *To find the volume of the tetrahedron OABC whose one vertex O is at the origin and the co-ordinates of the remaining three vertices A, B and C are* (x_1, y_1, z_1), (x_2, y_2, z_2) *and* (x_3, y_3, z_3) *respectively.*

Let **a**, **b**, **c** be the position vectors of the vertices A, B, C respectively w.r.t. O as origin. Since the co-ordinates of A are (x_1, y_1, z_1), therefore, the position vector **a** of the point A is given by

$$\mathbf{a}=\vec{OA}=x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}.$$

Similarly $\mathbf{b}=\vec{OB}=x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$ and $\mathbf{c}=\vec{OC}=x_3\mathbf{i}+y_3\mathbf{j}+z_3\mathbf{k}$.

Now the volume V of the tetrahedron $OABC$ is given by

$V=\frac{1}{6}[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ [See § 1 (A), for complete proof deduce this result here]

or
$$V=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad ...(5)$$

The formula (5) is the required formula.

(C) *To find the volume of the tetrahedron whose vertices have* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *and* (x_4, y_4, z_4) *as co-ordinates.*

Let A, B, C, D be the vertices of the tetrahedron $DABC$. Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the co-ordinates of the points A, B, C, D respectively. Then the position vectors of A, B, C, D are $x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k}$, $x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k}$, $x_3\mathbf{i}+y_3\mathbf{j}+z_3\mathbf{k}$, $x_4\mathbf{i}+y_4\mathbf{j}+z_4\mathbf{k}$ respectively.

We have $\vec{DA}=(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})-(x_4\mathbf{i}+y_4\mathbf{j}+z_4\mathbf{k})$

$=(x_1-x_4)\,\mathbf{i}+(y_1-y_4)\,\mathbf{j}+(z_1-z_4)\,\mathbf{k}$,

$\vec{DB}=(x_2-x_4)\,\mathbf{i}+(y_2-y_4)\,\mathbf{j}+(z_2-z_4)\,\mathbf{k}$,

and $\vec{DC}=(x_3-x_4)\,\mathbf{i}+(y_3-y_4)\,\mathbf{j}+(z_3-z_4)\,\mathbf{k}$.

∴ The volume V of the tetrahedron $DABC$ is given by

$V=\frac{1}{6}[\vec{DA}, \vec{DB}, \vec{DC}]$ [See § 1 (A)]

$$=\frac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 \end{vmatrix}$$

$$=\frac{1}{6}\begin{vmatrix} x_1-x_4 & y_1-y_4 & z_1-z_4 & 0 \\ x_2-x_4 & y_2-y_4 & z_2-z_4 & 0 \\ x_3-x_4 & y_3-y_4 & z_3-z_4 & 0 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Adding 4th row to 1st, 2nd and 3rd rows, we get

$$V=\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \quad ...(6)$$

The formula (6) is the required formula.

Corollary. Condition for four points to be coplanar.

The four points A, B, C, D will be coplanar if the volume of the tetrahedron formed by them is zero, *i.e.*

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0.$$

§ 2. *To find the volume V of a tetrahedron, in terms of the lengths of three concurrent edges and their mutual inclinations.*

Let $OABC$ be the tetrahedron. Take the vertex O as the origin. Let the lengths of the three edges OA, OB, OC be a, b, c and the angles BOC, COA, AOB be λ, μ, ν respectively. Let any three perpendicular lines through the origin O be taken as co-ordinate axes.

Again let the direction cosines of the lines OA, OB and OC be l_1, m_1, n_1, l_2, m_2, n_2 and l_3, m_3, n_3 respectively, so that the co-ordinates of the vertices A, B and C are (l_1a, m_1a, n_1a); (l_2b, m_2b, n_2b) and (l_3c, m_3c, n_3c) respectively. We have

$\vec{OA}=\mathbf{a}=l_1a\mathbf{i}+m_1a\mathbf{j}+n_1a\mathbf{k}$, $\vec{OB}=\mathbf{b}=l_2b\mathbf{i}+m_2b\mathbf{j}+n_2b\mathbf{k}$,

and $\vec{OC}=\mathbf{c}=l_3c\mathbf{i}+m_3c\mathbf{j}+n_3c\mathbf{k}$.

Again
$$\left.\begin{array}{l} \mathbf{a}\cdot\mathbf{b}=ab\,(l_1l_2+m_1m_2+n_1n_2)=ab\cos\nu \\ \mathbf{b}\cdot\mathbf{c}=bc\,(l_2l_3+m_2m_3+n_2n_3)=bc\cos\lambda \\ \mathbf{c}\cdot\mathbf{a}=ca\,(l_3l_1+m_3m_1+n_3n_1)=ca\cos\mu \end{array}\right\}$$

or
$$\left.\begin{array}{l} l_1l_2+m_1m_2+n_1n_2=\cos\nu \\ l_2l_3+m_2m_3+n_2n_3=\cos\lambda \\ l_3l_1+m_3m_1+n_3n_1=\cos\mu \end{array}\right\}. \qquad \ldots(1)$$

Now the volume V of the tetrahedron $OABC$ is given by

$$V=\tfrac{1}{6}\,[\mathbf{a},\mathbf{b},\mathbf{c}]=\tfrac{1}{6}\begin{vmatrix} l_1a & m_1a & n_1a \\ l_2b & m_2b & n_2b \\ l_3c & m_3c & n_3c \end{vmatrix}$$

$$=\tfrac{1}{6}\,abc\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}.$$

$$\therefore\ V^2=\frac{1}{36}a^2b^2c^2\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\times\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$=\frac{1}{36}a^2b^2c^2\begin{vmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_2l_1+m_2m_1+n_2n_1 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_3l_1+m_3m_1+n_3n_1 & l_3l_2+m_3m_2+n_3n_2 & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$=\frac{1}{36}a^2b^2c^2\begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix} \quad \text{[using the relations (1)].}$$

$$\therefore\ V=\pm\tfrac{1}{6}\,abc\begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{\frac{1}{2}}.$$

The negative sign will be neglected in calculating the magnitude of the volume V.

§ 3. *To find the volume V of the tetrahedron when equations of its four faces are given.*

Let the equations of the planes representing the four faces of the tetrahedron be

$$a_1x+b_1y+c_1z+d_1=0 \qquad \ldots(1)$$
$$a_2x+b_2y+c_2z+d_2=0 \qquad \ldots(2)$$
$$a_3x+b_3y+c_3z+d_3=0 \qquad \ldots(3)$$
$$a_4x+b_4y+c_4z+d_4=0. \qquad \ldots(4)$$

Now a set of any three planes out of the four planes given above, will intersect in a point, a vertex of the tetrahedron. Hence the four planes, taking three at a time, will intersect in ${}^4C_3$ i.e. 4 points, the four vertices of the tetrahedron.

Now solving (2), (3) and (4) by the method of determinants, we get

$$\frac{x}{\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}}=\frac{z}{\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}} \qquad \ldots(5)$$

Suppose
$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Let the capital letters represent the co-factors of the corresponding small letters in the determinant Δ, *i.e.* $A_1, B_1, C_1, D_1, A_2, B_2, \ldots$etc. represent the co-factors of $a_1, b_1, c_1, d_1, a_2, b_2, \ldots$ etc. respectively in Δ. The result (5) may be written as

$$\frac{x}{A_1}=\frac{-y}{-B_1}=\frac{z}{C_1}=\frac{-1}{-D_1}.$$

$\therefore$ The point of intersection of the planes (2), (3), and (4) is $(A_1/D_1,\ B_1/D_1,\ C_1/D_1)$. Similarly solving the other three sets of three planes, the points of intersection *i.e.* the other three vertices of the tetrahedron are