SCATTERING

THEORY

→ ## Theory of Scattering

The phenomenon of a beam of particle incident on a 'rest target' and deflected from their path, is called SCATTERING.

In scattering there is a collision of incident beam with target. Such collision may be elastic or non-elastic. In case of elastic collision, Total energy incident beam before collision is equal to that after collision. But in the case of non-elastic collision, energy before collision is greater than that from after collision.

SCATTERING CROSS SECTION:-

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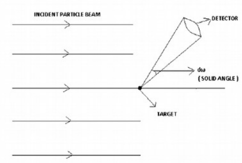
SCATTERING CROSS SECTION:-



When a beam of mono-energatic falls on a target consisting large number of scattering centres the particles are scattered in all directions and received by detector in a particular direction.

Then it was found – $dN \propto n|J_0|d\omega$..........(1)

Where - n = No. of scattering centres

$d\omega$ = Solid angle makes by detector at target

dN = No. of particles received by detector in unit time

J_0 = Incident flux (No. of particles that coming per unit area perpendicular to the direction of beam propagation in per unit time)

$dN = \sigma(\omega)n|J_0|d\omega$............(2)

The proportionality constant $\sigma(\omega)$ has the unit of area so, it is called scattering cross section.

" The area of cross-section of incident beam contains that are received by detector with given solid angle."

$$\sigma(\omega) = \frac{dN}{n|J_o|d\omega}$$

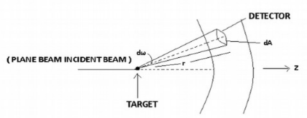
if $J_s(\omega)$ is the scattered flux in the solid angle ω then-

$$dN = |J_s(\omega)|r^2 d\omega$$

$$\sigma(\omega) = \frac{|J_s(\omega)|r^2 d\omega}{n|J_o|d\omega} \qquad \sigma(\omega) = \frac{|J_s(\omega)|r^2}{n|J_o(\omega)|}$$

The integeral of scattering cross-section over the solid angle is called total scattering cross-section. $\sigma_t = \int\sigma(\omega)d\omega$

QUANTUM MECHANICAL DESCRIPTION OF SCATTERING CROSS-SECTION



Let a beam of particles incident on target and scattered in all directions then by quantum mechanics, incident beam associated a plane wave and scattered particle associated a spherical wave so, the total wave function –

$$\varphi = \text{Plane wave function + spherical wave function}$$

$$\varphi = e^{ikz} + f(\theta,\Phi)\frac{e^{ikr}}{r}$$

Where $f(\theta,\Phi)$ is the scattering amplitude,

$$\text{Total incident flux} = |e^{ikz}|^2 \times \text{velocity of particle}$$

$$= |e^{ikz}|^2 \times v_1 = 1 \times v_1 = v_1 \text{ ....(2)}$$

$$\text{Total scattering flux (Per unit area per sec)} = |f(\theta,\Phi)|^2 \left|\frac{e^{ikz}}{r}\right|^2 \times v_2$$

$$= |f(\theta,\Phi)|^2 \times \frac{v_2}{r^2}$$

Suppose scattered particles are received by a detector at distance r from the target having solid angle dω.

$$d\omega = \frac{dA}{r^2} \qquad dA = r^2 d\omega$$

Thus the number of particles received by detector per second –

$$= \text{SCATTERING FLUX AREA}$$

$$= |f(\theta,\Phi)|^2 \frac{v_2}{r^2} \times dA$$

$$= |f(\theta,\Phi)|^2 \frac{v_2}{r^2} . r^2 d\omega.$$

$$= |f(\theta,\Phi)|^2 v_2 d\omega. \text{ ....(3)}$$

The differential scattering cross-section –

$$= \frac{\text{No.of particles recieved by detector per unit time}}{\text{Total incident flux}}$$

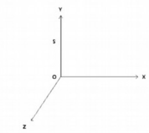
$$\sigma(\omega)d\omega = \frac{|f(\theta,\Phi)|^2 v_2 d\omega}{v_1}$$

In scattering, the velocity remains same – $v_1 = v_2$

$$\sigma(\omega)d\omega = |f(\theta,\Phi)|^2 d\omega$$

C- FRAME OF REFERENCE & L-FRAME OF REFERENCE

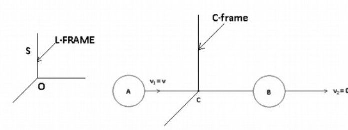
A co-ordinate system i-e used to describe the position & motion of any particle is called frame of reference. it is represented by -



If we take two particles A and B, where B is at rest and A is moving (travelling) towards B.

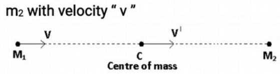
A co-ordinate system in which particle A is initially at rest is called Laboratory frame of reference. that means above velocities are measured with respect to L-frame of reference.

A co-ordinate system in which the centre of mass of two particles is initially and always at rest is known as C-frame of reference. (centre of mass for) that means co-ordinate system is taken on the centre of mass.



RELATION B/W ANGLES IN L-SYSTEM & C-SYSTEM :-

Let two particles with mass m_1 and m_2 in which m_2 is at rest but m_1 moves towards m_2 with velocity " v "



Then velocity of centre of mass is found as –

$$M_1 v + M_2 \times 0 = (M_1 v + M_2).v^{I}$$

$$V^{I} = \frac{M_1}{M_1+M_2}.V \qquad \text{..........(1)}$$

The velocity of particles relative to centre of mass -

$$V_1^{II} = V - V^{I} \qquad \text{[velocity of } M_1 \text{ w. r. t. C. M.]}$$

velocity of M_2 relative to C. M. ;

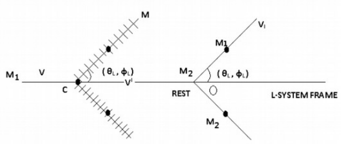
velocity of M_2 relative to C. M. ;

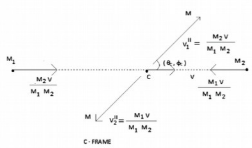
$$V_2^{II} = 0 - V^{I} \qquad V_1^{II} = V - \frac{M_1 v}{M_2+M_1}$$

$$V_1^{II} = \frac{M_2 V}{M_2+M_1} \text{.............(2)}$$

$$V_2^{II} = -\frac{M_1 V}{M_2+M_1} \text{.............(3)}$$

If the collision is elastic then speed remains the same before and after collision



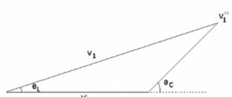


Before collision,

TOTAL MOMENTUM OF TWO PARTICLES = 0

After collision both particles definitely moves in opposite direction to make total momentum zero after collision as collision is elastic.

So - $\frac{M_1 M_2 v}{M_1+M_2} - \frac{M_1 v}{M_1+M_2} \times m_2 = 0$



From fig.

$$v_1 \cos\theta_L = v_1^{II} \cos\theta_C + v^{I} \text{ ....(4)}$$

$$v_1 \sin\theta_L = v_1^{II} \sin\theta_C \text{ ....(5)}$$

equation (5)/equation (4)

$$\tan\theta_L = \frac{v_1^{II} \sin\theta_c}{v_1^{II} \cos\theta_c + v^{I}}$$

$$\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + \frac{v^{I}}{v^{II}}}$$

$$\tan\theta_L = \frac{\sin\theta_c}{\cos\theta_c + \gamma} \qquad \textbf{where- } \gamma = \frac{v^{I}}{v^{II}}$$

Where – v^{I} = velocity of the C-frame relative to L-frame & v_1^{II} = velocity of M_1 particle relative to C-frame

<u>RELATION B/W SC ATTERING CROSS-SECTION IN L-SYSTEM & IN C-SYSTEM</u>

The number of particles scattered into given solid angles will remain equal in the both L & C-frame $(dN)_L = (dN)_C$

$$\sigma[\theta_C,\phi_L] dw_L = \sigma[\theta_C,\phi_C] d\omega_C \text{.......(1)}$$

Put , dw_L = = solid angle = $2\pi \sin\theta_L \, d\theta_L \, d\phi_L$

$$dw_L = = 2\pi \sin\theta_C \, d\theta_C \, d\phi_C$$

Using in equation number 1

$$\sigma[\theta_L,\phi_L] 2\pi \sin\theta_C \, d\theta_C \, d\theta_L = \sigma[\theta_C,\phi_C] 2\pi \sin\theta_C \, d\theta_C \, d\phi_C$$

$$\sigma[\theta_L,\phi_L] = \sigma[\theta_C,\phi_C] = \frac{\sin\theta_C \, d\theta_C \, d\phi_C}{\sin\theta_L \, d\theta_L \, d\phi_L}$$

For symmetrical collision – $\phi_C = \phi_L$

$$\sigma[\theta_L,\phi_L] = \sigma[\theta_C,\phi_C] = \frac{\sin\theta_C \, d\theta_C \, d\phi_C}{\sin\theta_L \, d\theta_L \, d\phi_L}$$

We know - $\tan\theta_L = \frac{\sin\theta_C}{\sin\theta_C + \gamma}$

$$\theta_L = \tan^{-1} \frac{\sin\theta_C}{\sin\theta_C + \gamma}$$

On differentiating

$$\frac{d\theta_L}{d\theta_C} = \frac{1}{\left(\frac{\sin\theta_C}{\cos\theta_C+\gamma}\right)^2} \left[\frac{(\cos\theta_C+\gamma)\cos\theta_C + \sin\theta_C^{\,2}}{(\cos\theta_C+\gamma)^2}\right]$$

$$= \frac{(\cos\theta_C+\gamma)^2}{(\cos\theta_C+\gamma)^2 + \sin\theta_C^2} \times \frac{Y\cos\theta_C + 1}{(\cos\theta_C+\gamma)^2}$$

$$= \frac{Y\cos\theta_C + 1}{(\cos\theta_C+\gamma)^2 + \sin\theta_C^2}$$

$$= \frac{Y\cos\theta_C + 1}{\sin\theta_C^2 + Y^2 + 2Y\cos\theta_C + \sin\theta_C^2}$$

$$\frac{d\theta_L}{d\theta_C} = \frac{Y\cos\theta_C + 1}{1 + Y^2 + 2Y\cos\theta_C}$$

On using in equation 2

$$\sigma[\theta_L,\phi_L] = [\theta_C,\phi_L] = \frac{\sin\theta_C}{\sin\theta_L}\left(\frac{1+Y^2+2Y\cos\theta_C}{Y\cos\theta_C+1}\right)$$

PARTIAL WAVE ANALYSIS :-

This method can be applied when the potential is central and having finite range. We know if motion is under central force angular momentum of particle is conserved.

Therefore if we categories the particles according to angular momentum then the scattering of particles of each angular momentum could be considered independent of the particles of other angular momentum.

But angular momentum contains a plane wave varies from zero to infinity thus it is possible to analyse a plane wave into infinite number of components each with definite angular momentum each component is called **partial waves** and process of decomposing a plane wave into partial waves is called **PARTIAL WAVE ANALYSIS.**

EXPANSION OF PLANE WAVE IN TERMS OF PARTIAL WAVE S (spherical waves)

(BAYER'S FORMULAE)

The plane wave e^{ikz} is a solution of free particle schrodinger's equation

$$H\psi(r) = E\psi(r)\ldots \qquad \ldots\ldots(1)$$

$$\text{With, } H = -\frac{\hbar^2}{2\mu}\nabla^2$$

there solution is $\psi(r) = R_{kl}(r).Y_{lm}(\theta,\phi)$ like hydrogen atom ....(2)

Where $R_{kl}(r)$ is the solution of radial part of schrodinger equation

If potential is absent $R_{kl}(r) = J_l(Kr)$ ....(3)

So equation (2) becomes $\psi = J_l(Kr).Y_{lm}(\theta,\phi)\ldots$ ....(4)

Where angular momentum quantum number $l = 0,1,2,3\ldots\infty$,

magnetic quantum number $m = -l \ldots\ldots +l$

Each ψ is called a spherical or partial waves. So,

$$e^{iKZ} = \Sigma_{l=0}^{\infty}\Sigma_{m=-l}^{+l} a_{lm}(r)J_l(kr)Y_{lm}(\theta,\phi) \qquad \ldots\ldots (5)$$

Where $a_{lm}(r)$ is the linear combination constant

If we assume propagation is along z-axis then K.r = Kr Cos θ = Kz

Which is independent on ϕ so **m = 0**

$$\text{so, } Y_{lo}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}}\,P_l(\text{Cos }\theta)$$

putting in eq (5) $e^{ikz} = \Sigma_{i=0}^{\infty} a_{lo}(r)J_l(Kr)\sqrt{\frac{(2l+1)}{4\pi}}P_l(\text{Cos}\theta)$ .... (6)

On collculating with the legendre

$$a_{l_o}\sqrt{\frac{2l+1}{4\pi}} = i^l(2l+1) \qquad \ldots\ldots\ldots\ldots(7)$$

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This is the bayrs formulae.

use asymptotic value of $J_l(kr)$ $\qquad J_l(kr) = \frac{\text{Sin}\left(kr-\frac{l\pi}{2}\right)}{kr}$

$$e^{ikr\cos\theta} = \frac{1}{kr}\Sigma_{i=0}^{\infty}(2l+1)i^l\text{Sin}\left(kr-\frac{l\pi}{2}\right)P_l(\text{Cos}\theta)$$

Where use – $\qquad \text{Sin } x = \frac{1}{2i}[e^{ix}-e^{-ix}]$

$$e^{ikr\cos\theta} = \Sigma_{i=0}^{\infty}\frac{(2l+1)i^l}{kr}\left[\frac{e^{i\left(kr-\frac{l\pi}{2}\right)}-e^{-i\left(kr-\frac{l\pi}{2}\right)}}{2i}\right]P_l(\text{Cos}\theta)$$

$$e^{ikr\cos\theta} = \Sigma_{i=0}^{\infty}\frac{(2l+1)i^l}{kr}\left[\frac{e^{i(kr)}.e^{-i\left(\frac{l\pi}{2}\right)}-e^{-i(kr)}e^{i\left(\frac{l\pi}{2}\right)}}{2i}\right]P_l(\text{Cos}\theta)$$

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Where $\qquad e^{-i\left(\frac{l\pi}{2}\right)} = [e^{-i\left(\frac{\pi}{2}\right)}]^l = (-i)^l$

$$e^{ikr\cos\theta} = \Sigma_{i=0}^{\infty}\frac{(2l+1)}{2ikr}[e^{i(kr)}+(-1)^{(l+1)}e^{-i(kr)}]P_l(\text{Cos}\theta) \qquad \ldots\ldots.(8)$$

The term $\frac{e^{-i(kr)}}{r}$ represents an incoming spherical wave while the term $\frac{e^{i(kr)}}{r}$ represents outgoing spherical waves.

PRESENT OF CENTERAL POTENTIAL V(r)

In this case $\qquad H = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r)$ ....(9)

Schrodinger eq. $\qquad H\psi(r) = E\psi(r)$

There sol

$$\psi(r) = \sum_{l=0}^{\infty}(2l+1)i^l R_{kl}(r)P_l(\cos\theta) \qquad \ldots\ldots\ldots\ldots(10)$$

But $R_{kl}(r)$ is the sol of radial part of schrodinger

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\left(K^2-U(r)-\frac{l(l+1)}{r^2}\right)\right]R_{kl}(r) = 0\ldots\ldots\ldots\ldots\ldots(11)$$

$$a_{l_0}\sqrt{\frac{2l+1}{4\pi}} = i^l(2l + 1) \qquad \text{............(7)}$$

Putting in equation (6) $e^{ikrCos\theta} = e^{ikz} = \sum_{i=0}^{\infty} i^l(2l+1)\, J_l(kr)P_l(Cos\theta)$

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